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On convergence of contractual trajectories in pure exchange economies

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The study is aimed to investigate the convergence to equilibrium of trajectories generated by contractual processes. “Contractual process” is a view on economy’s self-regulation, describing any state of exchange economy as a set of barter contracts among agents/coalitions. It is assumed that any moment of continuous time agents can partially break some contracts and sign more beneficial ones. Such reconstructing is called “benevolent”, when an agent/coalition breaks old contracts only when exhausting all other opportunities to increase welfare. Such processes are shown to converge to equilibrium under reasonable conditions, whereas non-benevolent processes need not converge, as shown by series of examples.

Keywords. Russia, core, contract, contractual allocation, contractual process (trajectory), tâtonnement, competitive equilibrium.

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NON-TECHNICAL SUMMARY

This project is purely theoretical, concerning our fundamental ideas about markets. We propose a new "contractual" approach describing economic behavior in disequilibrium situations. It develops a new version of market dynamics, adding one more approach to explain market functioning.

At least five different competing approaches have been elaborated so far to explain how real markets operate, and how economy attains equilibrium, namely: Walrasian tâtonnement, the disequilibrium dynamic model, Edgeworth's processes, Smale-type processes and Strategic bargaining approach. Each has some advantages and shortcomings, but none is completely successful and convincing. The famous historian of economic thought, Mark Blaug commented on this puzzle in an interview for "Challenge" (May-June 1998). To the question "What are the major issues on which we have not made progress?" he replied: "Markets and how they actually function; that is, how they adjust to match demand and supply. We in economics know a hell of a lot about equilibrium, but we really don't know how markets actually get to equilibrium." We would add that, in reality, we hardly see equilibrium itself, but rather some infinite convergence process involving reaction to shocks. This project suggests a sixth approach to this puzzle.

The basic idea is that in disequilibrium situations, under imperfect information, an agent can adjust both prices and quantities to disparities in demand and supply. Moreover, she often trades with a limited group of other agents that can be looked upon as "coalition." In different submarkets or coalitions, prices or "terms of trade" can be different, at least in disequilibrium. The notion of "contracts" enriches the classical concept of the "core" with dynamic details of making and breaking the personalized agreements and coalitions. The market process can then be looked upon as series of barter exchanges among emerging and declining agents' groups. A new concept of contractual process and its related contractual trajectory express this idea. We study several reasonable types of contractual trajectories, corresponding to different behavioral hypotheses. Some of them do converge to Walrasian equilibrium.

Informally, a contractual process describes an infinite horizon, continuous time trajectory. In each time agents possess, trade and consume some renewable resources. At any moment, any agent can exchange goods within many groups of agents (coalitions): she does not belong entirely to any coalition. Any coalition has a contract, which is a plan of exchange. The summation of endowments and all these trading (barter) contracts is the (current) consumption bundle for the agent. In the next instant an agent can renew old contracts, or break some of them to look for a better ones. The procedure is repeated many times, and it may result in stable barter contracts. Though there are no money and no explicit prices, it turns out that this stable situation is a usual competitive equilibrium, with uniform rates of exchange (supporting market prices). The question is whether this equilibrium can be reached through a converging contractual process under reasonable assumptions. A positive answer would make the whole concept logically compelling, supplementing, or even replacing the classical views on "tâtonnement" and on the generation of market equilibrium.

The results of our analysis show that, under very general assumptions, the contractual processes may or may not converge to equilibria. A series of examples, where non-convergence is caused by different reasons, is provided. The most interesting positive results were obtained for so called "*benevolent*" processes that converge to equilibrium under reasonable assumptions. It

is assumed that a coalition or an agent at first searches for new beneficial contracts among those offers that do not involve breaking other existing contracts. Only when that is impossible does the agent/coalition initiate new contracts, implying breaking, or decreasing in volume, old ones (*i.e.* the proportions of exchange remaining). This assumption requires rather benevolent and well-informed agents. However it is not too unrealistic to assume careful investigation of new possibilities before breaking the old contracts. It may also capture the impact of social institutions generating "trust" and "honesty" in business dealing.

There are some specific processes in microeconomics that appear similar in operation to this vision, namely "double auctions." There are also trade environments, like stock and commodity exchanges, in which agents, as assumed here, adjust price and quantity simultaneously without an auctioneer. Each agent suggests a volume (lot) to sell or buy and its (ask/bid) price. These decentralized offers and contracts drive the market. Perhaps this contractual approach can help clarify our understanding of the dynamics of such systems, thereby finding practical application.

A more ambitious goal is to apply this approach to macroeconomics. Maybe the economy as a whole operates more like a double auction, than like a classical Walrasian tâtonnement. Then the contractual process could better explain the stability or instability of markets, and their dynamic reaction to shocks, a primary concern of monetary macroeconomics. This approach could then make a contribution to the government regulation theory.

1. INTRODUCTION

Q. What are the major issues on which we have not made progress?
A. Markets and how they actually function; that is, how they adjust to match demand and supply. We in economics know a hell of a lot about equilibrium, but we really don't know how markets actually get to equilibrium.
From *Interview with Mark Blaug* (1998), Challenge, May–June

Modern models of economy, just as classical ones (Arrow-Debreu model or its simplest version, a pure exchange economy) are modelling processes of production and allocation of goods and are based on the concept of competitive equilibrium. At the same time, the proper mechanism of market functioning (how are prices settled or how do individuals, having chosen preferred consumption bundles, transit to final resource allocation?) still is not completely clear. In fact, classical presentation is that equilibrium prices are realized as a result of little by little permanently going *tâtonnement* process, which corrects current prices in accordance with excess demand law: price for a commodity increases if demand exceeds supply; when supply exceeds demand price decreases. Economic intuition says us that moving in this manner economic system as a whole has to find, to grope toward equilibrium prices. Applying mathematical terms this means that if one describes price change process by differential equation (inclusion), having in right hand side excess demand, then every solution of this equation converges to equilibrium prices[1]. However what is this demand and can we observe it in reality? In mathematical model by definition demand is the summation of optimal individual solutions in consumer problems, which are defined by current non-equilibrium prices and by agents' preferences. How is it possible to observe demand under *non-equilibrium* prices, if it is the sum of *unrealized* wishes to buy commodity bundles? One can observe the total volume of purchases or the volume of sellings, supply and its excess, but we think that demand is, evidently, fundamentally unobservable category.[2] Moreover, classical view on prices change in accordance with excess demand rule is commonly based on a fictitious auctioneer hypothesis. This auctioneer conducts prices, but he/she is not a revealed economic agent, more likely this is an impersonal being, realizing a market power.

In the modern literature are also available other approaches, different from classical tâtonnement, aimed on modelling of dynamics of market processes and the analysis of their convergence to equilibrium: processes of the prices changes, using Jacobi matrix of excess demand function (Smale's approach and other); disequilibrium models of trade (Hahn's process, Fisher's approach and other); Edgeworth's processes etc. The following section contains an extensive review of the literature on this theme, where comparative analysis of the approaches and directions is presented. However all approaches have the shortcomings, partially the same as in Walrasian tâtonnement (auctioneer and other), partially new, as, for example, high information requirement of processes with Jacobian and others. So, it allows to make a conclusion: classical

[1] This result holds only under additional strong assumptions (gross substitutability and etc.), however up to this moment it is not important.

[2] If one knows supply and excess supply is *positive*, then demand for a commodity can be calculated, but what can be done when excess demand is positive?

and other modern views on the market and laws of its homeostasis are not quite satisfactory from the modern point of view.

However can we suggest a constructive idea to solve the problem of attainability of economic equilibria and for better understanding of dynamics of market processes? To answer these questions we seemingly need to reconsider our views of what really occurs in the market. In our opinion there are a lot of commodity exchange dealings and for all involved individuals these dealings are mutually beneficial at the moment of their realization. The current resource allocation is generated as a summation of all the accomplished dealings and of an initial endowment allocation.

During a time some new dealings are realized, some of them reiterate there made earlier, other ones do not (probably this can be treated as a form of the rejection of signed in the past dealings which are non-beneficial at the moment). It is extremely important, that such "natural" process of a barter exchange goes itself, there is not here presented neither demand with the supply, nor prices. The investigation is aimed to formally describe and to study properties of these processes. Idea of the barter bargain by no means new in theoretical economics (*e.g.*, in Edgeworth's views), but it usually appeared as an interpretation, in the form of net trades in a formal model (see a survey below). Problem however namely in adequate formally-mathematical description of barter process, allowing an opportunity of a refusal from the bargains (breaking of the contract).

For formalization of suggested point on market functioning we propose to apply contract based approach. The formal theory of the contracts starts to be developed from seminal Makarov's papers (Makarov, 1980, 1982) and due to Kozyrev's results (Kozyrev, 1981, 1982) in which a key idea of partial breaking of the contracts has been offered. Hereinafter the theory of the contracts was essentially reconsidered and advanced in Marakulin (2003). Contractual approach is closer to an intuitive imagination on real processes of forming of prices and consumed resources and to deliver better understanding of cooperative and individual features of agents' behavior in a market. In particular, applying contractual approach one can suggest the clearer description of transition processes to stable (non-dominated) allocations and reveal a specific cooperative tâtonnement process, which formally-mathematical description is one of goals of this study. This cooperative tâtonnement supposes that coalitions of agents are able to sign new mutually beneficial contracts (exchange commodity dealings) and also each agent can partially break contracts signed in the past if it is beneficial for him. The process of the signing of new and breaking of old contracts is going in simultaneous regime (although, it is possible to consider the separate version), and is extended over time. The last means, that in fact process deals with momentary contracts, which together with signed in the past define the process derivative.

Formally contractual process can be described via differential inclusion $\dot{x}(t) \in F(x)$, where $x(t)$ is current allocation of resources. The right hand side of this inclusion is formed via mutually beneficial contracts for various coalitions where abilities of singleton coalitions are realized by means of partial breaking of contracts.

All feasible solutions of this inclusion form a set of feasible contractual trajectories, which can or cannot converge to (potentially) final allocations. It is known from the theory of contracts (Marakulin, 2003) that under some assumptions (interior point, differentiable concave utilities) every proper contractual allocation (this is an allocation which can be realized by a web of contracts stable relative to the signing of new contracts and relative to partial breaking of old ones) is equilibrium allocation. The converse implication is always true: every equilibrium can be presented as a proper contractual allocation. Thus equilibria and only they (under assumptions) are *stationary points* for cooperative proper contractual tâtonnement. However up to this moment it is *unknown* when this process, starting at initial endowments, is *converging* and which stationary points are *stable*. Exactly the investigation of convergency of contractual processes and related questions is the main goal of this project.

The main difficulties of our study, as for determination and so for stating of process convergence, are caused by an opportunity of the individuals partially to break contracts (because along a trajectory utilities may change non-monotonically), however it reflects the substance of market processes and otherwise equilibrium relative to initial endowments cannot be attained. It is also to analyze stability of equilibria, but the form of stability actually depends on the type of processes and it may differ for classical and cooperative tâtonnement. Interesting their comparison, and it is possible, that there are equilibria which are stable in one sense but unstable in another one. As a whole the analysis of convergence and stability of process also seems to be a necessary step coming into being the theory of barter contracts, this is important for economic theory and rather complicated problem.

2. SURVEY OF THE LITERATURE ON EQUILIBRIUM STABILITY

There is a vast economic literature, devoted to the research of processes driving a multiproduct economy to competitive equilibrium. By now one can mark out at least five approaches to explain market dynamics, they have own comparative advantages and shortcomings. These approaches are:

(i) Tâtonnement processes of equilibrium prices of Walrasian[3] type. This is tâtonnement, where a current disequilibrium prices change by the law of excess demand: if it is positive the price increases if negative then price decreases.

(ii) Processes, in which the law of change of prices is defined due to Jacobi matrix (differential) of excess demand function. The first process of this type was suggested in Smale (1981).

(iii) Disequilibrium models of trade processes among consumers; among them Hahn–Negishi process (Hahn, Negishi, 1962) and Edgeworth processes by Uzawa (Uzawa, 1962).

(iv) Edgeworth processes. They are the processes of commodity exchange without prices, they are based on a mutually beneficial barter (irrevocable) among the members of any coalition of consumers. As time elapsed the coalitions of agents participated in exchange may vary and run some class of permitted coalitions (some coalitions can be forbidden, may be because of that the formation of them is incredible from the essential point of view and exchanges are not realized).

(v) Strategic approach, where equilibrium and competition are examined from purely game theoretical point of view.

Below we consider the specified approaches in more details in (limited) frameworks of well-known Arrow–Debreu type *economy of pure exchange*. For convenience of an exposition we

[3] In this process market prices on different goods are changed simultaneously, this is a modification of original Walrasian idea suggested in Samuelson (1941).

begin with formal descriptions of model, introduction of notations and reminder of concepts and notions.

Let us consider a typical exchange economy in which $E = \mathbb{R}^l$ denotes the *space of commodities* (l is the number of commodities). Let $\mathcal{I} = \{1, \ldots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i = E_+ = \mathbb{R}^l_+$, an initial endowments $\omega_i \in X_i$, and a preference relation described by a utility function $u_i : X_i \to \mathbb{R}$, where $u_i(x_i) > u_i(y_i)$ means that agent i strictly prefers a bundle x_i to y_i. This may be also standardly denoted as $x_i \succ_i y_i$. So, the pure exchange model under study may be represented as a triplet:

$$\mathcal{E} = \langle \mathcal{I}, E, (X_i, u_i(\cdot), \omega_i)_{i \in \mathcal{I}} \rangle.$$

A pair of vectors (x, p), $x = (x_1, \ldots, x_n) \in \prod_{\mathcal{I}} X_i$, $p \in \mathbb{R}^l$, is said to be a *Walrasian or competitive equilibrium* of model $\mathcal{E}$, if the *price vector* $p \neq 0$ and the following conditions are satisfied:

(i) $\forall\, i \in \mathcal{I},\ px_i \leq p\omega_i\ \&\ \forall\, y_i \in X_i,\ \ y_i \succ_i x_i \Rightarrow py_i > p\omega_i$;

(ii) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \omega_i$.

Condition (i) says that for each i the bundle x_i is an optimal budget-admissible consumption plan, while (ii) means that these plans can be jointly realized (all markets are balanced). Traditionally the concept of competitive equilibrium usually express in the terms of equality between demand and supply or simply as the equality to zero of excess demand.

Individual demand $d^i(p)$ of agent i for prices $p = (p_1, \ldots, p_l) \neq 0$ is the solution of utility maximization problem

$$u_i(y) \to \max, \quad \textit{subject to } py \leq p\omega_i, \quad y \in X_i.$$

If $p \gg 0$ this problem always has a solution and therefore the map $d^i(\cdot)$ is well defined on $\mathbb{R}^l_{++}$. Together with individual demand it is sometimes convenient to consider *individual excess* demand function, determined by formula $z^i(p) = d^i(p) - \omega_i$.

In a context of assumptions for Arrow-Debreu model the map (function) of excess demand for a given prices is defined as *a sum* of individual solutions of consumer problems for all agents of economy (total demand) minus the total supply. In such a way the vector $D(p) = \sum_{\mathcal{I}} d^i(p)$ is called *total* or *aggregated* demand. As soon as in a context of exchange model supply is fixed and equal to $\bar{\omega} = \sum_{\mathcal{I}} \omega_i$, then *excess demand* under the prices p is

$$Z(p) = D(p) - \bar{\omega} = \sum_{\mathcal{I}} d^i(p) - \sum_{\mathcal{I}} \omega_i = \sum_{\mathcal{I}} z^i(p).$$

Obviously, the maps of demand and excess demand are correctly defined on area of change of the prices $\mathbb{R}^l_{++}$. Moreover, a vector $p \gg 0$ is the prices of equilibrium if and only if $Z(p) = 0$ (or, equivalently, $Z(p) \leq 0$). In general, the excess demand map can be a point-to-set mapping,[4] however for simplicity we shall assume below that demand is single-valued.[5]

Besides under the natural model assumptions (classical convex, continuous preferences) function $Z(\cdot)$ is continuous on its domain $\mathbb{R}^l_{++}$. Moreover, where it is necessary, we shall assume without

[4] For example, in Polterovich, Spivak (1982) it was investigated convergence to equilibrium of Walrasian processes and its stability under point-to-set excess demand mapping.

[5] It is provided by strictly quasi-concave utilities.

special mentioning that it is differentiable in a proper degree. By construction, the function of excess demand $Z(p)$ is homogeneous of degree 0 and satisfies to Walras law[6]: $\langle p, Z(p)\rangle = 0$, $\forall\, p \gg 0$. For the existence of equilibrium and also to provide the convergence of a number of processes of price changes to equilibrium, in addition it is usually assumed *boundary conditions*. The basic example of this kind condition is the following: if $p^m \to p^0 \in \partial\mathbb{R}^l_+$, $p^0 \neq 0$ for $m \to \infty$, then $Z_j(p^m) \to +\infty$ for $p^0_j = 0$, $j = 1, \dots, l$. In the limits of this section we always shall think that it is true everywhere, where it is necessary.

Further we proceed to the direct description of specified above processes and results.

2.1. Walrasian tâtonnement process of equilibrium prices

There is the vast literature, devoted to the study of this type processes, now we note only two reviews Hahn (1982), Polterovich, Spivak (1982), where one can find the detailed description of problems and results. Further first of all we describe process formally. The economy is described in most aggregated form via to excess demand function $Z : p \to Z(p)$, defined for all positive prices $p \gg 0$, $p \in \mathbb{R}^l$.

In a general case 'tâtonnement' is described as a process of prices $p(t) = (p_j(t))_{j=1,\dots,l}$, $t \geq 0$ changes, as the solution of the following system of the differential equations:

$$\dot{p}_j(t) = F^j(Z_j(p)), \quad j = 1, \dots, l. \tag{2.1}$$

Here it is always assumed that all functions $F^j(\cdot)$ are *sign-preserving*, *i.e.*, we have $Z_j(p)F^j(Z_j(p)) > 0 \iff Z_j(p) \neq 0$. Exactly this property of right hand side of (2.1) implies that the price of j-th commodity adjust in the same direction as excess demand for that commodity: it increases when excess demand is positive and vice versa. It is commonly assumed in addition (Hahn, 1982) that $\frac{d}{dx}F^j(x) > 0$ for points from an appropriate area for process (2.1). It is known (Hahn, 1982) that the solution of our system exists and is unique for every initial data $p(0) \gg 0$. In classical tâtonnement it is presumed a simplest form of functions $F^j(Z_j(p)) = Z_j(p)$, *i.e.*, in classic approach $F^j(\cdot)$ is identity map of real numbers into itself. In this case the system of differential equations (2.1) turns into $\dot{p} = Z(p)$. There are also considered some "intermediate" variants where, for example, it is supposed $F^j(Z_j(p)) = k_j Z_j(p)$ for some real $k_j > 0$, $j = 1, \dots, l$.

Essential treatment of tâtonnement process (2.1) is commonly based on a hypothesis of fictitious auctioneer, which, likely as it occurs in real auctions, raises the price when demand exceeds supply and, accordingly, reduces price, if the demand is less than supply. An auctioneer is not revealed in model (Arrow-Debreu type) economic agent, this is some impersonal being, whose actions reflects an invisible hand of the market. It is not quite clear in fact how the markets really work in a disequilibrium situation, during to search of equilibrium, because there are no presented revealed *microeconomic* models of process and of agents' behavior out of equilibrium. In the literature a hypothesis about the existence of auctioneer and similar constructions are criticized and are recognized as unrealistic, for example, see Kreps (1990) p. 195–198; Fisher (1983), p. 19–26. Fisher (Fisher, 1983) have noted the following difficulties in the process (2.1) interpretation: First, "... It has nothing to do with the question of whether or not trade, consumption, or production takes place out of equilibrium" (actually, exchange is possible only when equilibrium is attained!). Second, "...we know very little about how individuals do or

[6] For Arrow–Debreu model it is provided due to local non-satiated preferences.

ought to behave when equilibrium is not presented; hence, the resort to an aggregate equation". Finally, "... in the unrealistic world of no trading out of equilibrium ... individuals take action to make their excess demand effective. ... they *can* take such action which ... implies that they have something of value which they can and do sell so as to have something to offer when they buy..." So, out of equilibrium individuals have to buy and sell to reveal their excess demand, for the price changes to be going, but actually they can do it only under equilibrium prices.

Summarizing, one can conclude that equilibrium theory needs an adequate dynamic theory, in which framework the process of equilibrium prices searching has to be revealed.

The subject of criticism is also rather rigorous conditions, under which one can guarantee the convergence of process (2.1) to equilibrium. Results on convergence and stability of process (2.1) mainly are based on the property of *gross substitutability* of excess demand functions, in various forms of generality, or just on the *axiom of revealed preferences* and others. Let us consider these assumptions in more details, see Mas-Colell, *et al.* (1995).

- A function[7] $Z(\cdot)$ has the property of *gross substitutability* (GS–property), if for any prices p' and p such that $p'_m > p_m$ is true for some m and $p'_k = p_k$ for $k \neq m$, then $Z_k(p') > Z_k(p)$ takes place for all $k \neq m$, $k = 1, \ldots, l$.

  For a *differentiable* function $Z(\cdot)$ the condition of gross substitutability takes the form $\partial Z_k(p)/\partial p_m > 0$, $\forall\, p \gg 0$, $\forall\, k \neq m$.

- An excess demand function $Z(\cdot)$ satisfies to weak *axiom* (WARP) of *revealed preference*, if for any couple of vector-prices, p and p',

$$Z(p) \neq Z(p') \;\;\&\;\; pZ(p') \leq 0 \quad \Rightarrow \quad p'Z(p) > 0$$

  takes place.

Being applied to (aggregated) excess demand function, the properties of gross substitutability and (weak) revealed preference are, in general, *non-equivalent* and rather strong requirements. However both of them have common important corollaries:

(i) The set of equilibrium prices is *convex*.

(ii) If p^* is equilibrium price, then $p^*Z(p) > 0$ for all $p \gg 0$ which are *not proportional* to p^*.

In particular, property (ii) allows easily to understand why process (2.1) is converged, if one of the specified conditions is carried out. Really, in the simplest case of (2.1), when $\dot{p} = Z(p)$, it is enough to differentiate by t the function of squared Euclidean distance between a current prices and equilibrium prices, $||p(t) - p^*||^2 = \sum_{j=1}^{l}(p_j(t) - p_j^*)^2$. By virtue of Walras law, we obviously have $\frac{d}{dt}||p(t) - p^*||^2 = 2(p(t) - p^*)\dot{p} = -2p^*Z(p) < 0$. So, we see that the distance between a current vector of prices and equilibrium ones decreases when time is going.

Finally we would like to mention one more classical condition, providing the local convergence of tâtonnement process, this is the property of *diagonal domination* of Jacobi matrix $D_pZ^-(p^*) = A$, without the last row and column, of excess demand function $Z(p)$ in a point of equilibrium p^*. Formally, diagonal domination means

$$\exists\, h = (h_1, \ldots, h_{l-1}) \geq 0 : \;\forall\, j \;\; h_j a_{jj} < -\sum_{k \neq j} h_k |a_{jk}|.$$

[7] In a context of an exchange model the gross substitutability of excess demand function and demand function are equivalent.

Gross substitutability implies this property, but the opposite is false. Moreover, in literature there are not known other examples of diagonal domination. The diagonal domination and other similar requirements, *e.g.*, see Theorem 1.7 in Hahn (1982), implies that eigenvalues of A have negative real parts and this provides the local stability of price adjustment process.

2.2. Processes of the prices change, using Jacobi matrix of excess demand function

First of all we would like to notice, that a large part of the critical remarks, made relative to Walrasian processes, can be also addressed to the processes of this type.

Smale (Smale, 1976) investigated the convergence of prices changes process, based on (global) Newton method,[8] which is usually applied to find a solution of system of the nonlinear equations. The process is determined as:

$$[D_p Z^-(p)]\dot{p} = -\lambda(p) Z^-(p). \tag{2.2}$$

Here $Z^-(p)$ is excess demand for all goods excepting (for example) the last one, and $D_p Z^-(p)$ is Jacobi matrix of excess demand function, excepting the last row and column.[9] It is supposed, that the *sign* of functions $\lambda(p)$, entering as a factor in the right part of (2.2),[10] coincides with the sign of $(-1)^{l-1}\det[D_p Z^-(p)]$. If $D_p Z^-(p)$ is non-singular matrix in the domain of p changes, then process (2.2) can be rewritten in an explicit form

$$\dot{p} = -\lambda(p)[D_p Z^-(p)]^{-1} Z^-(p).$$

Smale (Smale, 1976) proved, that this process converges to equilibrium for any aggregated excess demand function,[11] if the initial prices $p(0) \neq 0$ are on the boundary of $\mathbb{R}^l_+$ (area of prices change), except for a set of zero measure (with the account of normalization), and under additional requirement that $D_p Z^-(p)$ is non-singular in effective area of prices change (these are positive and normalized by the last component). Certainly, this is remarkable result, however its weak side is too large informational requirements. Really, at each time moment the process of prices change requires a knowledge not only excess demand, but also Jacobi matrix, *i.e.*, the change of price in the market explicitly depends on how the prices on other markets are changed.

Kamiya (Kamiya, 1990), developed Smale's approach, and has offered process, defined as

$$\left[\frac{D_p Z^-(p)}{||D_p Z^-(p)||} - \frac{I}{||p - p(0)||}\right]\dot{p} = -\lambda(p) Z^-(p). \tag{2.3}$$

Here, as well as in Smale's process, $D_p Z^-(p)$ is Jacobi matrix of excess demand without the last column and row, $p(t)$ is $(l-1)$-dimensional vector-function of the prices *without* last component

[8] For the first time the method was offered and partially investigated in Arrow, Hahn (1991).

[9] To the last row and column there corresponds a commodity, which is used as a numeraire good. The elimination of a row and column is necessary, to allocate square nonsingular submatrix in $J[Z(p)]$; since excess demand is homogeneous, $J[Z(p)]p = 0$ and, therefore, matrix $J[(p)]$ is always singular.

[10] Certainly, it is necessary also to postulate other properties of $\lambda(p)$, ensuring existence and uniqueness of solution (2.2).

[11] Here the economy *completely* is set by function $Z(p)$.

$p_l(t)$, such that $||p(t)|| \leq 1$ and $p(0)$ is an initial vector of the prices of the same dimension. It is assumed the *sign* of real-valued function $\lambda(p)$, placed in the right part of (2.3), is *opposite* to the sign of the determinant of matrix, entered in the left part of (2.3), *i.e.*, the sign coincides with the sign of $\det \left[\frac{I}{||p-p(0)||} - \frac{D_p Z^-(p)}{||D_p Z^-(p)||}\right]$. Using methods, suggested by Smale, Kamiya (Kamiya, 1990) proves, that the process (2.3) converges to equilibrium for almost all initial data $p(0)$ from the *interior* of $\mathbb{R}^{l-1}_+$.

Mukherji (Mukherji, 1995) investigated another process, using Jacobi matrix of aggregated excess demand function:

$$\dot{p} = -J[Z(p)]^t Z(p). \tag{2.4}$$

He has shown in Mukherji (1995), that the process (2.4) belongs to a group of so-called locally effective processes (LEPM): the processes of this type converge to any (regular[12]) equilibrium locally (*i.e.* for the equilibrium prices there exists a neighborhood, such that if $p(0)$ in the neighborhood, the process converges to the equilibrium).

Concerning all described above processes, and also other processes from this group,[13] it is possible to state one common remark: all of them require too much information. Moreover, Saari and Simon (Saari, Simon, 1978) have proved, that this is unavoidable property of any LEPM-process, *i.e.*, actually, it is necessary condition for the process to be locally effective for (almost) any function of aggregated excess demand. In relationship we would like to recall Sonnenschein-Debreu-Mantel results, see survey Shaher, Sonnenschein (1982), about representation of a general aggregated excess demand function as an excess demand function for Arrow-Debreu model. They show, that any continuous, homogeneous and obeying Walras law function allows representation in the specified for Arrow-Debreu model form, the model where number of the agents is equal to the number of commodities. In so doing the utilities of individuals may be classical: continuous, strictly concave, monotonous and, moreover, homogeneous (degree 1). Thus, one can go to the following conclusion: *any* locally *effective mechanism of prices change* based on excess demand function for Arrow-Debreu model is *informational requiring* and, with necessity, has to use (whole!) Jacobi matrix $J[Z(p)]$ of excess demand. In particular, making comments to Smale's process, Hahn (Hahn, 1982, p. 767) replies: "Obviously these results are interesting as algorithms and not as models of invisible hand."

2.3. Disequilibrium models of trade processes

There are at least two disequilibrium processes known in literature in a context of a pure exchange model, see Fisher (1983), Mukherji (2003). These are Edgeworth process by Uzawa (Uzawa, 1962), and Hahn's process (Hahn, Negishi, 1962), so-named in Negishi (1962). It is common feature of both processes, that *endowments* are varied at the time, *i.e.*, initial endowments $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^{ln}_+$ is the *function of time*, $\omega : [0, +\infty) \to \mathbb{R}^{ln}_+$ (n is the number of agents). As well as in Walrasian processes (tâtonnement), real consumption comes only at the end of the process, where it is described by a limiting point of $\omega(\cdot)$. Further we consider other specific features of processes.

Common properties. The prices change according to excess demand:

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[12] Jacobi matrix of excess demand at equilibrium point has the maximal rank equal to $l - 1$.

[13] For example, for orthogonal Newton process, described in Jordan (1983).

$$\dot{p}_j(t) = \begin{cases} F^j(Z_j(p,\omega(t))), & \text{unless} \quad p_j = 0 \ \& \ Z_j(p) < 0; \\ 0, & \text{if} \quad\quad p_j = 0 \ \& \ Z_j(p) < 0. \end{cases} \tag{2.5}$$

Here the functions $F^j(\cdot)$ satisfy to the usual requirements: continuity and sign-preservation.

There is postulated a *law of change* of the initial endowments, which can be also treated as (current) allocation of consumed resources $\omega : [0,+\infty) \to \mathbb{R}^{ln}_+$, and this map has to obey the requirements:

$$\forall\, i \in \mathcal{I}, \ \ \dot{\omega}_i(t) = g_i(p(t),\omega(t)) - \omega_i(t), \ \ \sum_{i=1}^{n} \omega_i(t) = \sum_{i=1}^{n} \omega_i(0), \ \forall\, t \in [0,+\infty), \tag{2.6}$$

where all functions $g_i(p(t),\omega(t))$ are assumed to be continuous and, in addition, to satisfy to "No Swindling" condition:

$$\forall\, i \in \mathcal{I}, \ \ p(t)\dot{\omega}_i(t) = 0 \iff p(t)g_i(p(t),\omega(t)) = p(t)\omega_i(t), \ \forall\, t \geq 0. \tag{2.7}$$

Essentially, in both processes the functions $g_i(p(t),\omega(t))$, $i \in \mathcal{I}$ set *a rule of trade* (trading or transaction rule). Other requirements in processes differ.

Hahn's process. A specific requirement is the assumption that markets are *orderly*: $\forall\, t \geq 0$

$$z^i_j(p(t),\omega_i(t))Z_j(p(t),\omega_i(t)) > 0, \ j = 1,2,\ldots,l, \tag{2.8}$$

unless the case $z^i_j(p(t),\omega_i(t)) = 0$ for all $i = 1,2,\ldots,n$. Here $z^i_j(p(t),\omega_i(t))$ is *individual excess* demand of i's agent for the commodity j under current prices and endowments $\omega_i(t)$. This requirement means, that if the market of a product j is not balanced, then *all* agents have positive excess demand or supply (there are hence only unsatisfied demanders or suppliers for any given good).

Edgeworth's process by Uzawa. It is supposed, that endowments (here it is a current consumption) are changing so that monotonous growth of utility of each individual goes, at least for one in strictly form, *if* it is *possible in general*; everything under constrains (2.6), (2.7). Formally it is defined as: $\forall\, t \geq 0$

$$\begin{gathered} u_i[g_i(p(t),\omega(t))] \geq u_i[\omega_i(t)], \ \forall\, i, \\ u_i[g_i(p(t),\omega(t))] = u_i[\omega_i(t)], \ \forall\, i \iff g_i(p(t),\omega(t)) = \omega_i(t), \ \forall\, i \ \& \\ \forall\,(x_1,\ldots,x_n) \in \mathbb{R}^{ln}_+ : p(t)x_i = p(t)\omega_i(t) \ \forall\, i \ \& \sum x_i = \sum \omega_i(t), \\ \exists\, k: \ u_k(x_k) > u_k(\omega_k(t)) \ \Rightarrow \ \exists\, i: \ u_i(x_i) < u_i(\omega_i(t)). \end{gathered} \tag{2.9}$$

Thus, at each current moment of time, a state of economy changes if and only if it appears possible mutually beneficial exchange within the framework of budget constrains.[14]

Both described processes (Hahn's process and Edgeworth's process by Uzawa) are converged to some Pareto optimal allocation under more or less standard assumptions, including boundary condition (provides a movement of a trajectory in $\mathbb{R}^{ln}_{++}$), and additional assumption about

[14] Uzawa also assumes, that functions $g_i(\cdot)$, determining barter process along a trajectory, take values equal to the demand of individuals, if aggregated demand is equal to supply (price of equilibrium for current initial endowments). I did not identify a place where we need it. In my opinion this is excessive assumption, though I cannot disagree with it.

strict concavity of utility functions, see Hahn, Negishi (1962), Uzawa (1962). In so doing price processes are also converged (for Edgeworth's process by Uzawa see Mukherji, 1974, 2003) and the limiting prices are the prices of equilibrium for the given limiting resources allocation (here this is initial and final allocation simultaneously), this is a situation of no-trade.

The detailed description of specified disequilibrium processes one can find in Arrow, Hahn (1991) (part 13), Hahn (1982), Fisher (1983), Mukherji (2003), where their criticism is also contained. For example, Fisher (Fisher, 1983), being an advocate of Hahn's process, criticized Edgeworth's process by Uzawa in the following way. First, it is not clear, why Pareto improving trade actually will takes place whenever such a situation arises. The reason is that it is possible that *all* coalitions of the agents, capable to do such mutually beneficial exchange, can have too large size, and small coalitions (bilateral or trilateral or quadrilateral trade and so on) are unable to carry out Pareto-improving exchange. Admitting an opportunity of exchange in the huge coalitions, we impose "very heavy requirements on the dissemination of information and to assume away the costs of coalition formation." Moreover, the inclusion into model of money as means of exchange, actually does not change a situation. Second, Edgeworth's processes do not admit a revealed opportunity of production and consumption in nonequilibrium situations, now it is still open quest.[15] Third, the assumption, that a trade takes place only when the utilities of individuals are increasing, is not as harmless as it seems. In a true nonequilibrium world the individuals "trade even then there is no direct utility (or profit) gain from so doing because they wish to take advantage of arbitrage opportunities, speculating on their ability later to retrade at more advantage prices." Moreover, Fisher writes: "Yet a crucial aim of stability theory must be to examine the question of whether arbitrage drives a competitive economy to equilibrium."

On the other hand, Hahn's process does not avoided shortcomings. One of them is that before purchasing something, one needs to sell something. Therefore many of the potentially interesting bargains may be not realized. In order to solve this problem, Arrow and Hahn (Arrow, Hahn, 1991) directly introduced money into model, using it as the intermediary-goods in any barter bargain, and imposed other additional assumptions. Mukherji (Mukherji, 2003) also criticizes Hahn's process and specifies a main its shortcoming: the absence of revealed *voluntary* nature of barter bargains in process.

Really, if there is no specific model, explaining in an microeconomic way how and why (non-mutually beneficial) barter bargain is realized, voluntary can be understood only as a condition, attracting monotonous growth of utility of the individuals along trajectory, *i.e.*, realizing bargain all its participants should win.

Finally, both processes are indirectly based on a hypothesis of auctioneer, since due to definitions they satisfy (2.5).

2.4. Edgeworth's processes

We call so the processes of change of current resources allocation, described in continuous or discrete time, which are going *without prices* and, accordingly, there are no budget constrains. Essentially the processes of this type are close to contractual processes without breaking of the

[15] Hahn (Hahn, 1982) notes the work of Hurwicz–Radner–Reiter (Hurwicz, *et al.*, 1978), in which the process is considered in a stochastic context and, moreover, there is also shown that the production can be incorporated into model.

contracts, see Hahn (1982) p. 772–777, and also following section. The basic sense of process is, that the process generates a trajectory in space of allocations such that along trajectory there is the monotonous growth of individual utilities, and at least for one strictly, if a current point still is not Pareto optimum. A process of this type can be set, for example, via a rule of trade, described by functions $g_i(\omega(t))$, $i \in \mathcal{I}$, similarly as it was made in the previous paragraph (however here there are no prices): $\omega : [0, +\infty) \to \mathbb{R}^{ln}_{+}$,

$$\forall\, i \in \mathcal{I}, \;\; \dot{\omega}_i(t) = g_i(\omega(t)) - \omega_i(t), \;\; \sum_{i=1}^{n} \omega_i(t) = \sum_{i=1}^{n} \omega_i(0), \; \forall\, t \in [0, +\infty).$$

The functions $g_i(\cdot)$ are continuous and satisfy to conditions (2.9), *with the exception* of budget constrains.[16] It is simply to prove that every limiting point of such process is Pareto optimal.[17]

There is a number of papers where Edgeworth's processes are considered in stochastic context (see review Hahn, 1982; and also Hurwicz, *et al.*, 1978, Graham, Weintraub, 1975), where, in our terms, on the set of all mutually beneficial contracts some reasonable probability distribution is defined. The appropriate stochastic process converges to an Pareto optimum with probability 1 (we omit other specific features and assumptions).

There are also papers, in which Pareto boundary is attained by the efforts of coalitions of limited size (agents are not more than number commodities). The first result of this type was received in Polterovich (1970), see also Feldman (1973), Graham, *et al.* (1976), Madden (1975), Green (1974). However, in so doing each active in barter process coalition carries out transition on intra-coalitional Pareto boundary (relative to current allocation) and all permissible coalitions are incorporated in a cycle, which is repeated infinite times (compare with (2.9)). Thus, essentially these processes are discrete in time.

There are also papers, in which the transition to core allocations are realized, see Green (1974). Here a stochastic context is also available, where the reaction of current blocking coalition replenishes with the reaction of supplementing coalition, that forms the transition from a current allocation in subsequent one. In such a way an allocation from core (if necessary the procedure is repeated infinitely) is attained with probability 1.

Easily to see, that in all of these directions is available contractual context and, moreover, the contract based language is simpler, more convenient and it can be better interpreted.

2.5. Strategic approach

This direction began to develop in the economic theory from the middle of 80's of last century and was aimed to clarify the basic hypotheses of competitive equilibrium theory in a context

[16] This assumption is important for the process converges to Pareto optimum. Hahn in Hahn (1982) p. 773, gives another description of process, requiring only a growth of utilities without restrictions on the derivative of process. Such process can be finished at an "irredundant" point, which is not Pareto optimal, an elementary example can be constructed: Let $\alpha : [0, +\infty) \to [0, 1]$, $\operatorname{supp}(\alpha) = [0, \infty)$, $\int_0^\infty \alpha(t)dt = 1$ (one can take $\alpha(t) = \frac{1}{(1+t)^2}$, $\int_0^t \alpha(s)ds = 1 - \frac{1}{(1+t)}$) and let v be a mutually beneficial contract such that $u_i(x_i + \lambda v_i)$ strictly increases in $\lambda \in [0, 1]$, $\forall\, i \in \operatorname{supp}(v)$, but $z^* = x + v$ is not Pareto optimum. Then a trajectory of process $z(t) = x + v \int_0^t \alpha(s)ds$ obeys all Hahn's conditions and $z(t) \to x + v$ for $t \to +\infty$.

[17] We can not give exact reference, but it seems that this (elementary) fact was clear to economists for a long time ago, and Uzawa is one of them.

of a strategic game.[18] An idea was to apply game theoretical methods to give the answers on such questions as: whence the prices are undertaken and who them defines, why the agents should accept the prices as given and why they cannot change them (a consumer is said to be a "price-taker"), that is equilibrium and perfect competition? The answers on these and other important theoretical questions are given in the analysis of some game in extensive form. These games belongs to a class of DMBG-games (dynamical matching and bargaining games), constructed by a model of economy in a special way. For lack of an opportunity to enter in detailed explanations, we specify only two sources of the literature: Gale (2000), Kunimoto, Serrano (2004) and we describe only the basic idea of the approach[19] in a context of one possible game model.

In economy there is a continuum of agents, presented by a finite number of types. Each type is characterized by initial endowments vector and by von Neumann-Morgenstern utility functions. Commodities are infinite divisible, time is discrete and is indexed by the natural numbers. At each time period each agent can meet the partner with some fixed probability. If some pair $\{i, k\}$ of the agents has met, then, after the identification of type and current consumptions, x_i and x_k, with equal probability one of them is selected, say i, to propose a vector of goods $z \in \mathbb{R}^l$ (makes *the offer*), to be transferred to him from his opponent k. If the agent k accepts the offer, her/his consumption bundle changes on $x_k - z$, and the partner's one becomes $x_i + z$ (only the allowable offers are under consideration, they are not to allow consumption bundles to leave the limits of consumption sets).[20] If the agent does not accept the offer, the consumptions do not change. The individual, who does not accept the offer made to him, can leave the market at next time moment, no other individual (accepting the offer or not participating in the given round) can leave market. An agent, who never leaves market, receives utility equal to $-\infty$ (thus, the consumption is possible only after leaving). A player's strategy is a plan the prescribes her/his bargaining behavior in different trade situations for each period, depending on current consumption, type of partner, her/his current consumption and, the offer made by him (if so happen). The strategy of the agent depends on the realized (by him) earlier bargains and, if to him the offer is made, it may takes values: "accept the offer", "reject and stay", "reject and exit".

It is supposed, that the agents of one type use common strategy. However, in view of the previous acts of trade, the different agents can have different current consumption bundles, but for each type only a finite number. Further, for constructed DMBG-game a concept market equilibrium is introduced, which is in fact a specialized kind of perfect Bayesian equilibrium and in discrete (equivalent) variant of game is presented as a sequential equilibrium. The basic result is the theorem, which states that in every market equilibrium each player leaves the market with probability 1, when her/his consumption bundle is equal to a bundle kept him in Walrasian equilibrium.

In spite of the fact that the methods of strategic approach are essentially different from contractual approach, we have found necessary to describe this direction, because it is aimed to solve similar problems of equilibrium theory, which are related with the validity and possibility to be correctly realize basic theoretical hypotheses. In our opinion, the most attractive part

[18] It is also called as a game in normal form.

[19] Apparently, it was Douglas Gale's idea, but it seems that it is a suitable adaptation and development of ideas of previous researchers, see Gale (2000).

[20] In our terminology the offer z is an offer to sign the barter contract $(z, -z)$.

of strategic approach is the clear description of what and how the individuals make trades in uncertain market circumstances.

3. DYNAMICAL CONTRACTUAL PROCESS AS COOPERATIVE TÂTONNEMENT

We begin with description of contractual economy and main contract based concepts in an appropriate form of generality. Further the basis of dynamical contractual processes is described.

3.1. Main concepts of contractual economy

Let us consider a typical exchange economy, described in section 2 and presented as a triplet:

$$\mathcal{E} = \langle \mathcal{I}, E, (X_i, u_i(\cdot), \omega_i)_{i\in\mathcal{I}} \rangle.$$

Let us denote by $L = E^n$ the space of economy allocations, let $\omega = (\omega_i)_{i\in\mathcal{I}}$ be the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i\in\mathcal{I}} X_i$ and define

$$\mathcal{A}(X) = \{x = (x_i)_{i\in\mathcal{I}} \in X \mid \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \omega_i\},$$

the set of all *feasible allocations* in $\mathcal{E}$.

Everywhere below we shall assume that model $\mathcal{E}$ satisfies the following *smoothness* assumption **(S)**.

(S) *All utilities $u_i(\cdot)$ are concave and twice continuously differentiable functions, such that $\forall\, x_i \in X_i = \mathbb{R}^l_+$, $\forall\, i \in \mathcal{I}$, $\nabla u_i(x_i) \neq 0$ and matrices $\nabla^2 u_i(x_i)$ are negative definite.*

Further we shortly recall different contractual concepts, see Marakulin (2003).

By the formal definition, any reallocation of commodities $v = (v_i)_{i\in\mathcal{I}} \in L$, where $v_i \in E$, $i \in \mathcal{I}$, *i.e.*, any vector $v \in L$ satisfying $\sum v_i = 0$, is called a (barter) *contract*. In this project context we assume that *every* contract is permissible.

A finite collection V of permissible contracts is called *a web of contracts relative to* $y \in \mathcal{A}(X)$ if

$$y + \sum_{v\in U} v \in X \quad \forall\, U \subset V.$$

A web of contracts V relative to ω is called *a web of contracts* or simply *a web*. Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(X)$. Notation $x(V) = \omega + \sum_{v\in U} v$ denotes the feasible allocation sustained by V relative to ω. For any contract $v \in V$, let us set

$$S(v) = \mathrm{supp}\,(v) = \{i \in \mathcal{I} \mid v_i \neq 0\}.$$

the support of the contract v. It is assumed that contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she simply may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, *i.e.*, as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$. It is required formally that each element $U \in F(V, T)$ has to satisfy the following properties:

(i) $v \in V \setminus U \;\Rightarrow\; S(v) \cap T \neq \emptyset$,

(ii) $v \in U \setminus V \;\Rightarrow\; S(v) \subset T$.

Condition (i) means that only members of T can break contracts in V, condition (ii) means that only members of T may sign new contracts.

In contract-based approach the notion of domination via a coalition is extended onto webs of contracts. This property of domination via coalition $T \subseteq \mathcal{I}$, being written as $U \underset{T}{\succ} V$ (read: U dominates V via coalition T), means that

(i) $U \in F(V, T)$,

(ii) $x_i(U) \underset{i}{\succ} x_i(V)$ for all $i \in T$.

A *web* of contracts V is called *stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$.

A *web* of contracts V is called *lower stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $U \subseteq V$.

A *web* of contracts V is called *upper stable* if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $V \subseteq U$.

An allocation x is called *contractual* (lower, upper contractual) if $x = x(V)$ for a stable (lower, upper stable) web V.

It can be directly deduce from definitions that in any standard market every core allocation allows an alternative description as contractual one; accordingly, Pareto optimal allocations correspond to upper contractual ones, and individual rational allocations are lower contractual ones etc. The concept of proper contractual allocation is also important, this concept realizes (assumptions: interior point, smooth preferences) an alternative description of equilibria.

The notion of proper contractual allocation is introduced due to the following construction. First let us introduce an equivalence relation on the set of all lower stable webs, this equivalence will allow us to partially divide contracts. To this end, let us define a partial ordering on the set of all webs as follows:

$$U \geq V \iff \exists \text{ a map } \textit{onto } f : U \to V, \text{ such that}$$

(i) $\lambda f(u) = u$ for some $0 \leq \lambda \leq 1$ and for every $u \in U$,

(ii) $\sum_{u \in f^{-1}(v)} u = v$ for every $v \in V$.

Thus, a relation $U \geq V$ simply means, that contracts from U are produced from the contracts from V due to partition into several contracts (decomposition in a sum) under condition of preservation of exchange proportions and volumes of exchanged commodities.[21]

Now the equivalence relation may be defined as follows:

$$U \simeq V \iff \exists \text{ a web } W \text{ such that } V \geq W \;\&\; U \geq W.$$

Definition 3.1. *An allocation x is called proper contractual if there exits a web V such that $x = x(V)$ and for every $U \simeq V$ the allocation $x = x(U)$ is contractual.*

[21] A value $f(u) \in V$ specifies the contract, a share of which is $u \in U$.

The economic meaning of proper contractual stability of an allocation is, that we allow the agents not only to sign new contracts but also to partially break contracts if exchange proportions remain constant. This extends agents' operating potentialities and approaches contractual processes to market processes under perfect competition conditions. In Kozyrev (1981, 1982), Kozyrev stated (see also Marakulin, 2003) that under some technical assumptions all proper contractual allocations are equilibria.

Further let us turn to the main subject of project. We suggest to investigate the stability of trajectories which correspond to the proper contractual behavior of traders. However before we would like to specify one possible interpretation of proper-contractual behavior, driving economy to proper-contractual allocations.

Suppose that an economy is not static and lives during long-duration interval of time. As time elapsed individuals sign the rather short-term contracts on an exchange of commodities. The contract assumes mutual deliveries of goods among agents and, after its execution, an opportunity of renewal, *i.e.*, the same contract can be signed again, but now it is realized during *another* time period. The agents can agree with contract's renewal (prolongation) or disagree, first studying an opportunity to prolong contract in smaller volumes. Thus, instead of *breaking of the contract*, even if partial, for economy in dynamics living a long time period one *can speak about renewal* and *non-renewal* of the contracts. Notice, that if resources are renewed then according to this interpretation agents can consume goods as time goes on, notwithstanding the fact that current situation is a disequilibrium one. It seems natural to assume that stable in time contracts, *i.e.*, regularly renewed contracts have to take out economy to equilibrium performance (there is no production!). However the convergence to such state is not clear and requires the careful research.

We believe that suggested description of economic exchange processes — due to rules of proper-contractual behavior — essentially closer to intuitive representations about their character in real economic environment in comparison with processes considered in items (i)–(iv) of the previous section.

3.2. On the definition of contractual trajectory

Formally, a trajectory is a map $x(\cdot)$, operating from $[0, +\infty)$ into the set of all feasible allocations, *i.e.*, into $\mathcal{A}(X)$,

$$x(\cdot) : [0, +\infty) \to \mathcal{A}(X).$$

Here the vector $x(t) = (x_i(t))_{\mathcal{I}}$ is a feasible bundle of consumption plans, realized at the moment $t \geq 0$. It is presumed that $t = 0$ is the initial time point, the process 'starts' at this point from initial endowments allocation, *i.e.*, we set $x(0) = \omega$.

We are interested in not arbitrary trajectories of this type, but trajectories which can be realized during contractual processes via commodity exchange among agents. Presume $\Delta t > 0$ is the time period during which a contract v is realized, and presume that other *exchange operations* with commodities (the signing of new contracts or the breaking of existing ones) *were not realized.* Then at the moment $t' = t + \Delta t$ trajectory takes value $x(t') = x(t) + v$, wherefore $v = x(t') - x(t)$. As soon as other contractual operations in interval $[t, t']$ were not conducted, one may think that at the point $t'' = \lambda t' + (1 - \lambda)t$, $\lambda \in [0, 1]$ the trajectory value is produced from values of end points, which are mixed in the same proportions, *i.e.*, one can postulate $x(t'') = \lambda x(t') + (1-\lambda)x(t)$.[22] This can be rewritten in the form $x(t'') = x(t) + \lambda v \Rightarrow x(t+\lambda\Delta t) - x(t) = \lambda v$ and therefore,

$$\dot{x}(t) = \lim_{\lambda\to+0} \frac{x(t+\lambda\Delta t) - x(t)}{\lambda\Delta t} = \frac{v}{\Delta t} \implies v = \dot{x}(t)\Delta t.$$

Further let us assume that during time interval $[t, t']$ there were a (finite) sequence of signed contracts, such that their *time periods* of realization *are not overlapping.* Let m be a number of contracts. One can think that the final time point of one contract is simultaneously the starting point of another contract: if not we can always replenish system with an appropriate number of zero contracts. So, interval $[t, t']$ is divided into m intervals, determined by points $t = t_0 < t_1 < \ldots < t_m = t'$, such that $[t_{k-1}, t_k]$ are time intervals of contracts $v_k = x(t_k) - x(t_{k-1})$ realization, $k = 1, \ldots, m$. Put $\Delta t_k = t_k - t_{k-1}$ and due to previous formula find

$$x(t') = x(t) + \sum_{k=1}^{m} v_k = x(t) + \sum_{k=1}^{m} \dot{x}(t_{k-1})\Delta t_k = x(t) + \int_t^{t'} \dot{x}(s)ds.$$

As soon as by assumption contractual process starts at the moment $t = 0$ at the point ω, we have

$$x(t) = \omega + \int_0^t \dot{x}(s)ds, \quad \dot{x}(s) = \frac{v_k}{\Delta t_k}, \ \forall\, s \in [t_{k-1}, t_k].$$

Further holding away ourself from the latter (simple) deduction or in other words, if we allow ourself to consider a limit variant of last formula then the number of contacts is passing to infinite and the realization time of each contract is passing to zero, one can do the following conclusions.

(i) *Contractual trajectory*, which for a finite number of contracts is represented as integral of some step function, in general case is the integral of some *integrable* on every finite interval function $\dot{x}(\cdot)$ and is defined via formula

$$x(t) = \omega + \int_0^t \dot{x}(s)ds. \tag{3.1}$$

In other words *contractual trajectory* is an *absolutely continuous*[23] on every interval $[0, t]$, $t > 0$ map

$$x(\cdot) : [0, +\infty) \to \mathcal{A}(X).$$

So, we have provided the first property of contractual trajectory definition.

(ii) *Derivative* $\dot{x}(\cdot)$ of contractual trajectory in general case is defined almost everywhere on $[0, +\infty)$ and the value $\dot{x}(t)$ defines a (momentary) contract, signed at the moment $t \in [0, +\infty)$. If the time $\Delta t > 0$ of contract realization is known, that formally means $\dot{x}(t') = \dot{x}(t'')$, $\forall\, t', t'' \in [t, t+\Delta t]$, the resulting (gross) contract can be found from $v(t) = \dot{x}(t)\Delta t$. In other words,

[22] Applying 'physical' interpretation, one can say that we postulate the uniform (constant) speed of contract realization in interval $[t, t']$.

[23] A function $f(\cdot)$ with domain $[a, b]$ is said to be absolutely continuous if $\forall\varepsilon > 0\ \exists\delta > 0$ such that $\Sigma_{k=1}^{m} |f(b_k) - f(a_k)| < \varepsilon$ holds for every finite system of pairwise non-overlapping intervals $(a_k, b_k) \subset (a, b)$, $k = 1, 2, \ldots, m$, which obeys $\Sigma_{k=1}^{m}(b_k - a_k) < \delta$.

the derivative of contractual trajectory can be understood as a barter contract per time unit. Notice also the obvious corollary: the *range* of derivative is the subspace of contracts, *i.e.*,

$$\dot{x}(\cdot) : [0, +\infty) \to L^c, \quad L^c = \{v \in L \mid v = (v_i)_{\mathcal{I}} : \sum v_i = 0\}. \tag{3.2}$$

One more remark in addition. What is a contract for given trajectory? By definition of contractual trajectory (curve) we can not determine it in general because we do not know the duration of contract's realization (what does mean zero duration?). This is why one can correctly say only about *momentary contracts*, or about the summation of contracts, signed during a non-zero time interval. Keeping this point of view, one can say about summation of contracts for a "measurable time" $\Omega \subseteq [0, \tau]$, where Ω is any measurable subset of interval. In such a case we have

$$\textstyle\sum_{\Omega} v(s) = \displaystyle\int_{\Omega} \dot{x}(s) ds.$$

Surely, items (i), (ii) do not describe all properties of contractual trajectory related with contractual processes; these are only initial, unconditional requirements. Further let us consider other features of trajectory which correspond to contractual processes. In addition it is necessary to take into account conditions, at which contracts are signed, and also character of a trajectory changes under the breaking of contracts.

For the constructive description of contractual processes, related with the breaking of contracts, it is convenient to consider extended understanding of a trajectory described below, we shall call this a *coalitional* trajectory.

Suppose that for each coalition $S \subseteq \mathcal{I}$ with at least two elements, $\mathrm{card}(S) \geq 2$, an (absolutely continuous) map

$$v^S : [0, +\infty) \to L^c_S, \quad L^c_S = \{v \in L \mid v = (v_i)_{\mathcal{I}} : \sum_{i \in S} v_i = 0 \ \&\ v_i = 0, \forall\, i \notin S\} \tag{3.3}$$

is determined. Essentially, $v^S(t)$ is gross (total) contract, achieved by the members of a coalition S at a moment $t \geq 0$. A collection of all such maps $\{v^S(t)\}_{S \in \mathbf{K}} = V(t)$, related with a set of *permissible* coalitions $\mathbf{K} \subset \mathbf{2}^{\mathcal{I}}$, obviously determines a trajectory in previous sense by formula

$$x(t) = \omega + \sum_{S \in \mathbf{K}} v^S(t), \quad t \geq 0. \tag{3.4}$$

Notice, that in this description of contractual trajectory we actually describe not only a current allocation, but a set of *varying with time contracts*, where each *coalition* has *the only* current gross contract (for forbidden coalitions — zero). As time elapsed this set can be transformed according to the rules of proper-contractual behavior. Therefore, to ensure that an allocation realized after partial breaking of the contracts from $V(t)$ is feasible, it is necessary in addition to require that $V(t)$ is a *web* of contracts (relative to ω).

Finishing we would like to note one important thing. Basically the trajectory $x(t)$, $t \in [0, +\infty)$ comprises the whole information on the contracts made by the agents, their volumes and the time moments of signing. This information is contained in the derivative of trajectory $\dot{x}(t)$. Therefore a coalitional-contractual trajectory is not a new object, but just a convenient form for representation of information in adequate aggregated kind. Really, for each map the value

$v^S(t)$ at a point $t \geq 0$ can be determined by the formula

$$v^S(t) = \int_{\Omega_S^t} \dot{x}(s)ds, \quad \Omega_S^t = \{s \in [0,t] \mid \dot{x}_i(s) \neq 0, \ i \in S \ \ \& \ \ \dot{x}_i(s) = 0, \ i \in \mathcal{I} \setminus S\}.$$

However there is one nuance here, which can appear in the case when two or more pairwise non-intersected coalitions are independently signed new contracts at the same moment of time, or simply the time intervals of contracts' realizations are overlapping. Essentially, the consideration of such situations is consistent, especially in coalitional-contractual context. To avoid some possible collisions, related with appearing now ambiguity in the restoration of gross coalitional contract via derivative of trajectories, an easiest way is to conduct analysis in the terms of a coalitional trajectory.

3.3. Contracting and recontracting processes

A coalition can sign a new contract only if all members of coalition have relevant motives in signing, *i.e.*, after contract's realization (up to current moment) the utility of every member has to increase. In last section we have seen that contract per time unit is the derivative of trajectory at time point. Thus for smooth preferences one can think that contract v will be signed by coalition S, *i.e.*, trajectory moves along vector $v = \dot{x}(t)$ only if $\operatorname{supp}(v) = S$ and

$$\langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle > 0, \quad \forall\, i \in S.$$

Since $v_i = 0$ if $i \notin S$, then we can write a determining condition:

$$\dot{x}_i(t) \neq 0 \ \Rightarrow \ \langle \dot{x}_i(t), \nabla u_i(x(t)) \rangle > 0, \quad \forall\, i \in \mathcal{I}, \ \forall\, t \geq 0. \tag{3.5}$$

This condition characterizes moment t as the case of contract's signing. Now let us consider the case of *contracts' breaking.*

The description of contractual process with the partial breaking of contracts is possible in rather general framework. However in such a case the formal-mathematical analysis of process, with the purpose to prove its convergence, looks very difficult, at least on this stage of research. This is why further we shall make a several simplifying hypotheses. These hypotheses determine basic parameters: which *contracts*, in which *time moment* and in which *volume* are broken off, *i.e.*, all vagueness of contractual process related with the breaking of contracts are revealed.

The decision on partial break of the contracts is accepted by each agent individually, in conditions of a sufficient information for myopic-rational breaking of the contracts. We conceive that, in difference with a signing of the new contract, where an individual needs to find the partners and to pass a stage of negotiations about the future contract, the breaking of contracts is simpler decision and, therefore, can be accepted and is realized without temporary delays, as soon as there is the suitable opportunity. This motivates the following hypothesis.

> **(IB)** *Instantaneous Breaking of the contracts.* In each time moment each individual instantly (for zero time) partially breaks the signed earlier contracts in optimum volume.

This hypothesis does not say anything about what contracts and in which volume can be broken off. In a general case pertinently to think, that each individual has an opportunity (right) to break in any volume any contract, signed earlier current time moment $t \geq 0$. However, to simplify the subsequent analysis, it is possible to consider for the beginning some particular cases, a little bit limiting opportunities on break of the contracts.

For the *aggregated* contractual trajectory, defined in (3.1), (3.2) we shall postulate:

> **(UB)** *Uniform Breaking of all contracts.* At each time moment each individual can partially break *all* contracts, signed in economy to the given moment, but just in *identical measure* (proportion).

This hypothesis assumes, that at a current moment $t \in [0, +\infty)$ each agent makes a decision on break of the contracts. This decision is based on minimum of the *information*, extracted only from *current allocation* $x(t)$ and not accepting in attention the values $x(t')$, "passed" by a trajectory in previous time moments $t' \in (0, t)$. Apparently, such sight on an opportunity to break contracts is acceptable for an economy with small number of agents, where it is possible to assume, that the contracts are signed only by a coalition of all agents (grand coalition). However if economy consists of many agents this assumption is problematic. Really, it is not clear why do effect of breaking of contracts with involved persons has to influence in such crucial manner — the break in the same measure — on uninvolved directly into contract individuals? However, to carry out break only for a part of the contracts in which agent is involved, that is better for essence of contractual process, it is necessary that this part to be explicit. One of simple variants of revealing this information is to consider the trajectory in coalitional-contractual form described in the previous section.

For *coalitional* trajectory, described in (3.3), (3.4) we shall assume:

> **(CUB)** *Coalitional Uniform Breaking of contracts.* At each moment of time each individual can partially break *all* contracts, signed by any coalition, in which she/he participates, and in limits of a coalition *in an identical measure*, but, probably, in different proportions for different coalitions.

From the informational point of view this hypothesis means, that each agent stores (remembers) the aggregated information about intra-coalition exchanges, in the form of "gross" contract. Thus, now the results of breaking of contracts by an individual will influence only the agents directly involved in the barter contract with this individual by means of gross coalitional contract, and it does not concern to exchanges in other coalitions. As the special case of this hypothesis, it is possible to examine variant when *breaking* and the *signing* of new contracts occurs in frameworks of *the same coalition.*

4. PROPER-CONTRACTUAL UB-PROCESSES: UNIFORM BREAKING OF CONTRACTS

In the previous section we have discussed contractual processes without breaking of contracts, and also have considered some properties and hypotheses, related with the partial breaking of contracts, signed earlier current time moment. Further we are going to consider processes, in which the signing and partial break of the contracts go in *simultaneous mode.*

At first, with the purpose to simplify the subsequent analysis, we shall consider the case of *aggregated* proper-contractual trajectory, for which it is admitted only partial breaking of *all* contracts signed to the current moment, and all in an equal measure.

Further we have to clarify that means the fact that an individual i at the moment t *did not want*, but after contract v signing, at some moment $\tau > t$ he/she *wants* to partially break contracts. The first means that $\langle \nabla u_i(x(t)), x_i(t) - \omega_i \rangle \geq 0$, the second one that $\langle \nabla u_i(x_i(t) + (\tau - t)v_i), x_i(t) + (\tau - t)v_i - \omega_i \rangle < 0$. Therefore, does exist a moment $t + \Delta t \in [t, \tau]$ such that

$$\langle \nabla u_i(x_i(t) + \Delta t v_i), x_i(t) + \Delta t v_i - \omega_i \rangle = 0.$$

Notice also, that by virtue of **(IB)** the effect of contracts' breaking can influence the change of a trajectory at a moment t if and only if in *each* neighborhood of t there is a moment $\tau > t$ with the specified above properties. Passing $\tau \to t$ we obtain

$$\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0, \tag{4.1}$$

and this is the first condition, which defines the motion of trajectory with contracts' breaking. Notice that the point defined by equation (4.1) is the maximal point of utility $u_i(y_i)$ on the ray starting at the point ω_i in direction $x_i - \omega_i$ (here $y_i = \omega_i + \lambda(x_i - \omega_i)$, $\lambda \geq 0$).

Further let us consider another condition. Primary, at the time t of signing, contract v was mutually beneficial at the point $x(t)$. The fact that at the moment $\tau > t$ an individual $i \in \operatorname{supp}(v)$ partially breaks gross contract $x(t) + (\tau - t)v - \omega$ in a volume $1 - \alpha$ means that from the point $x(t)$ the trajectory moves to point $z = \omega + \alpha(x(t) + (\tau - t)v - \omega) = (1 - \alpha)\omega + \alpha(x(t) + (\tau - t)v)$, $0 \leq \alpha < 1$. Once again this is a maximum point for agent i's utility on the linear segment, linking $x(t) + (\tau - t)v$ with initial endowments vector (thus there is a projection along straight line going through two points). Therefore new point of trajectory

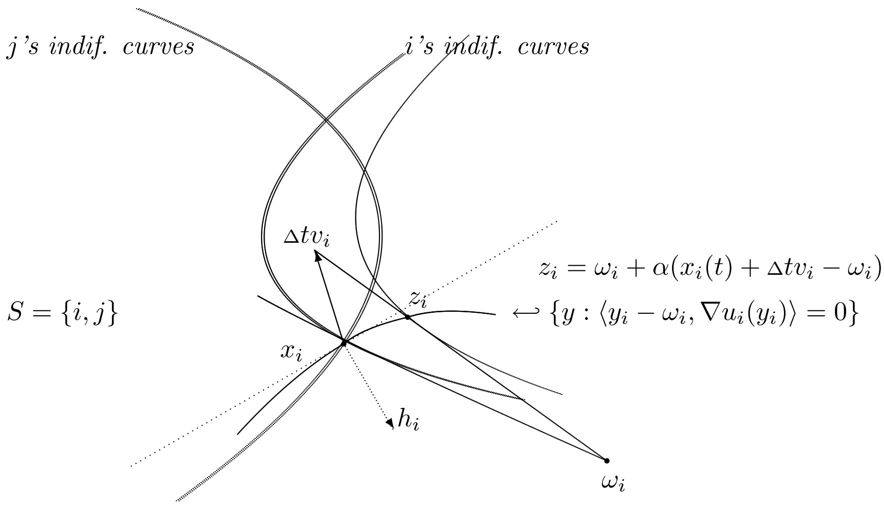

Fig. 1. Proper-contractual transition

has to satisfy to equation

$$\langle \nabla u_i(x_i(\tau)), x_i(\tau) - \omega_i \rangle = 0.$$

Setting $\Delta t = \tau - t$ and substituting in equation expressions

$$x_i(t + \Delta t) = x_i(t) + \Delta t \dot{x}_i(t) + o(\Delta t),$$

$$\nabla u_i(x_i(t+\Delta t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(\Delta t\dot{x}_i(t) + o(\Delta t)) + o(\Delta t\dot{x}_i(t) + o(\Delta t)),$$

which are true due to Taylor's formula,[24] and taking into account (4.1), we find

$$\Delta t\langle \nabla u_i(x_i(t)), \dot{x}_i(t)\rangle + \Delta t\langle \nabla^2 u_i(x_i(t))\dot{x}_i(t), x_i(t) - \omega_i\rangle +$$
$$\Delta t^2\langle \nabla^2 u_i(x_i(t))\dot{x}_i(t), \dot{x}_i(t)\rangle + o(\Delta t) = 0.$$

Now one can divide this on Δt and pass to limit over $\Delta t \to 0$. As a result we are coming to equation

$$\nabla u_i(x_i(t))\dot{x}_i(t) + \langle \nabla^2 u_i(x_i(t))\dot{x}_i(t), x_i(t) - \omega_i\rangle = 0 \iff$$
$$\langle h_i(x_i(t)), \dot{x}_i(t)\rangle = 0, \qquad h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \omega_i). \tag{4.2}$$

Equations (4.1), (4.2) describe important properties of contractual trajectory but still do not completely define process. It is necessary also take into account the dependence of $\dot{x}(t)$ from initially mutually beneficial contract v, signing which agents are coming to beneficial breaking of contracts for one of individuals. The situation is illustrated in Fig. 1 (in Edgeworth box style), which reflects a character of transition and objects involved into analysis.

Recall that from the point $x(t)$ trajectory moves on to the point $x(t+\Delta t) = \omega + \alpha_i(x(t)+\Delta tv - \omega)$, $0 \le \alpha_i < 1$ at the moment $t + \Delta t$. In general the value α_i depends on current consumption $x_i(t)$, (momentary) contract v and duration $\Delta t > 0$ of its realization. By virtue of assumption **(S)** the model is smooth, and it is easily to see, that $\alpha_i(x, v, \Delta t)$ is *differentiable* function (in general locally), implicitly defined from the equation

$$\langle \nabla u_i(x_i(t+\Delta t)), x_i(t+\Delta t) - \omega_i\rangle = 0, \quad x_i(t+\Delta t) = (1-\alpha_i)\omega_i + \alpha_i(x_i(t) + \Delta tv_i). \tag{4.3}$$

Here parameter $\alpha_i \ge 0$ determines a point $(1-\alpha_i)\omega_i + \alpha_i(x_i(t) + \Delta tv_i)$ of i's utility maximum on the line, parameterized as: $\omega_i + \lambda(x_i(t) + \Delta tv_i - \omega_i)$, $\lambda \ge 0$. If $\alpha_i < 1$, then at a point $x_i(t) + \Delta tv_i$ the breaking of contracts is realized in volume $1 - \alpha_i$, and, there is no the break if $\alpha_i \ge 1$.

From representation $x_i(t+\Delta t)$ in the right part of (4.3) we have

$$\frac{x_i(t+\Delta t) - x_i(t)}{\Delta t} = \frac{(\alpha_i(x, v, \Delta t) - 1)}{\Delta t}(x_i(t) - \omega_i) + \alpha_i(x, v, \Delta t)v_i,$$

whence, passing to a limit on $\Delta t \to 0$, with the account $\alpha_i(x(t), v, \Delta t)_{|\Delta t=0} = 1$ (by virtue of (4.1)), we obtain

$$\dot{x}_i(t) = \lambda_i(x_i(t) - \omega_i) + v_i, \quad \lambda_i = \frac{\partial \alpha_i(x(t), v, \Delta t)}{\partial \Delta t}|_{\Delta t=0}.$$

Further, value λ_i is possible to find from the equation (4.2),

$$\langle h_i(x_i(t)), \lambda_i(x_i(t) - \omega_i) + v_i\rangle = 0 \quad \Rightarrow \quad \lambda_i = \frac{\langle h_i(x_i(t)), v_i\rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t))\rangle}. \tag{4.4}$$

Thus, if at a moment t there exists *only one* agent, satisfying to (4.1), with number i, the trajectory locally will change under the law

$$\dot{x}(t) = \lambda_i(x, v)(x(t) - \omega) + v, \quad \lambda_i(x, v) = \frac{\langle h_i(x_i(t)), v_i\rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t))\rangle}.$$

[24] $o(\cdot)$ is the standard notation of infinitesimal value.

Moreover, presented considerations allow to reveal complete conditions, describing a moment t as a situation of break of the contracts at a current allocation $x(t)$ and when (momentary) a contract v is signing. The break is realized if the individual i satisfies (4.1), value $\alpha_i(x(t), v, \Delta t)$ is locally *decreased* in Δt at a point $\Delta t = 0$, *i.e.*, if derivative with respect to Δt is negative. So, for breaking it is necessary and enough[25] that $\lambda_i(x, v) < 0$. Further, by virtue of the assumption **(S)** the matrix of second partial derivatives $\nabla^2 u_i(x_i(t))$ is negatively defined, whence by virtue of (4.1) and (4.2) for $x_i(t) - \omega_i \neq 0$ we conclude

$$\langle h_i(x_i(t)), \omega_i - x_i(t)\rangle = -\langle \omega_i - x_i(t), \nabla^2 u_i(x_i(t))(\omega_i - x_i(t))\rangle > 0.$$

Thus, denominator in (4.4) is always positive and, therefore, the situation of breaking of contracts by the individual i is completely characterized by condition (4.1) and additional condition

$$\langle h_i(x_i(t)), v_i\rangle < 0.$$

What will take place in the case of several agents, desiring to break off contracts, when a new contract v starts to be realized? In other words, how will the contractual process go, if *more than one* individual satisfies (4.1)? For all these individuals values $\alpha_i(x(t), v, \Delta t)_{|\Delta t=0} = 1$, but the character of process is determined by their derivatives. It is clear, that the break will happen only if *at least one* derivative relative to Δt is negative, and the measure of breaking is defined by greatest absolute value from negative derivatives. Thus, it is proved the following

Lemma 4.1. *Consider contractual process with partial breaking of barter contracts, satisfying to hypotheses* **(IB)**, **(UB)** *— Instant Uniform Breaking of all contracts. Let (x, v) be a couple achieved in process at a moment $t \geq 0$, where $x = x(t) = (x_1, \ldots, x_n)$ is allocation and $v = (v_1, \ldots, v_n)$ is a momentary mutually beneficial barter contract, signed among individuals at the moment t. A pair (x, v) sets a situation of breaking of contracts if and only if for some $i \in \mathcal{I}$ takes place*

$$\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i\rangle = 0 \quad \& \quad \langle h_i(x_i(t)), v_i\rangle < 0, \tag{4.5}$$
$$h_i(x_i(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(x_i(t) - \omega_i).$$

In such case the local law of contractual process is defined by equation

$$\dot{x}(t) = \lambda(x, v)(x(t) - \omega) + v, \tag{4.6}$$

where $\lambda(x, v)$ is minimum of $\lambda_i(x_i, v_i)$, calculated for individuals $i \in \mathcal{I}$, satisfying condition $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i\rangle = 0$; here

$$\lambda_i(x_i, v_i) = \frac{\langle h_i(x_i(t)), v_i\rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t))\rangle}.$$

As it already was noted, if the break of contracts does not occur, the local law of change of a contractual trajectory is set by a rule $\dot{x}(t) = v$. Hence, one can apply the law (4.6) in a general case, if for $\lambda(x, v) > 0$ replace this value by zero. Combining this fact with the result of previous Lemma 4.1, we come to the following definition of proper-contractual trajectories. Let's define

$$\lambda^{min}(x, v) = 0 \bigwedge \min\left\{ \frac{\langle h_i(x_i), v_i\rangle}{\langle h_i(x_i), (\omega_i - x_i)\rangle} \;\middle|\; i : \langle \nabla u_i(x_i), x_i - \omega_i\rangle = 0 \right\}.^{26} \tag{4.7}$$

[25] When $\lambda_i(x, v) = 0$ the point $x = x(t)$ can be or to not be a limit point of breaking contracts points of a trajectory.

[26] Here standardly $a \wedge b = \min\{a, b\}$.

Definition 4.1. *An absolutely continuous map $x(\cdot) : [0, +\infty) \to \mathcal{A}(X)$ is called proper contractual trajectory under hypotheses* **(IB)**, **(UB)**, *if the following conditions are satisfied*:

(i) $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle \geq 0$, $\forall\, i \in \mathcal{I}$;

(ii) *Derivative of the trajectory obeys the law*

$$\dot{x}(t) = \lambda^{min}(x, v)(x(t) - \omega) + v, \tag{4.8}$$

where $v \in L^c$ is mutually beneficial contract, i.e., $\langle \nabla u_i(x_i(t)), v_i \rangle > 0$, $\forall\, i \in \operatorname{supp}(v)$, *and value $\lambda^{min}(x, v)$ is defined by* (4.7).

Notice, that due to this definition proper contractual trajectory is actually described as a solution of some differential inclusion

$$\dot{x}(t) \in F(x), \quad x(0) = \omega$$

on interval $[0, +\infty)$, where the right hand side obeys (i), (ii).

The law of change of proper contractual trajectory (4.8) can have another form, but, certainly, the law should to satisfy restrictions (4.1), (4.2). Really, (4.8) postulates the certain form of projection of current allocation $x = (x_1, \ldots, x_n)$ on area, defined by constrains

$$\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle \geq 0, \quad \forall\, i \in \mathcal{I}.$$

Essentially, this projection corresponds to a procedure of partial breaking of all contracts, and the area, on which allocation is projected, is the set of all allocations stable relative to the partial breaking of all contracts (or, alternatively, realized by a proper-contractual web $V = \{x - \omega\}$, see Marakulin, 2003). In doing so the projection is realized along the ray $(\omega_i - x_i)$, where i is the individual, satisfying (4.1), onto hyperplane $\{z \in \mathbb{R}^l \mid \langle h_i(x_i), z \rangle = \langle h_i(x_i), x_i \rangle\}$, where the vector $h_i(x_i)$ is determined by (4.2). Such kind of projection assumes, that only contracts signed before the moment t may be broken, and that the contract v is untouched. However it is possible to postulate another law of breaking, in which the breaking of current new contract is possible:

$$\dot{x}(t) = \beta(x, v)v + (1 - \beta(x, v))(\omega - x(t)). \tag{4.9}$$

This process can be treated as a break with delay, *i.e.*, the break occurs only after realization of (momentary) contract v. A value $\beta(x, v)$ can be found from the equation (4.2), whence, if i satisfies (4.1), we find

$$\beta(x, v) = \beta_i(x_i(t), v_i) = \frac{\langle h_i, x_i(t) - \omega_i \rangle}{\langle h_i, x_i(t) - \omega_i \rangle + \langle h_i, v_i \rangle}. \tag{4.10}$$

Notice, that numerator in (4.10) is always negative and that the denominator is a sum of numerator with the value $\langle h_i, v_i \rangle$, which in a case of contracts' breaking is negative, and, therefore, if the break of the contracts is beneficial for i, the whole ratio is a value between zero and unit. The value $\beta_i(x_i(t), v_i) \leq 1$ defines for process (4.9) a measure of contracts' breaking: when it is less the break is more. Moreover, for $\beta_i(x_i(t), v_i) \geq 1$ should be $\langle h_i, v_i \rangle \geq 0$ and the break does not occur. In case when several individuals satisfy (4.1) the break should come true at a maximum level. Therefore, the law of a trajectory change has the following form. Let's define

$$\beta^{min} = \beta^{min}(x(t), v) = 1 \wedge \min\{\beta_i \mid i \in \mathcal{I} : \langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = 0\},$$

where values $\beta_i(x(t), v)$ are set by the formula (4.10). Then our process obeys the law

$$\dot{x}(t) = \beta^{min} v + (1 - \beta^{min})(\omega - x(t)), \tag{4.11}$$

where $v \in L^c$ is a mutually beneficial contract, *i.e.*, $\langle \nabla u_i(x_i(t)), v_i \rangle > 0$, $\forall\, i \in \mathrm{supp}\,(v)$.

How described processes correspond among themselves? In essence they are equivalent. Really, if only one agent is ready to break contracts and carries out it in process (4.8) then locally this law can written as

$$\dot{x}(t) = \frac{\langle h_i(x_i(t)), v_i \rangle}{\langle h_i(x_i(t)), (\omega_i - x_i(t)) \rangle}(x(t) - \omega) + v = \frac{1}{\beta_i}[\beta_i v + (1 - \beta_i)(\omega - x(t))],$$

where β_i is defined by (4.10). It is easy to see that in general case (a few agents satisfy (4.1)) will be the similar connection, *i.e.*, we have

$$\dot{x}(t) = \lambda^{min}(x, v)(x(t) - \omega) + v = \frac{1}{\beta^{min}(x, v)}[\beta^{min}(x, v)v + (1 - \beta^{min}(x, v))(\omega - x(t))].$$

Thus, the distinction in processes is reduced to some positive factor, defined by function $\beta^{min}(x, v)$, that is graded by the arbitrariness in a choice of mutually beneficial contract v.[27]

5. PROPER-CONTRACTUAL CUB-PROCESSES: COALITIONAL-UNIFORM BREAKING OF CONTRACTS

Further we shall consider the concept of *coalitional* proper-contractual trajectories, *i.e.*, a trajectory, satisfying to hypothesis **(CUB)** (instead of **(UB)**). To do it we can apply a significant part of the above analysis, but there are also essential differences:

(1) The partial breaking of gross intra-coalitional contract for different coalitions can be realized in a different degree (measure).

(2) Breaking the gross coalitional contract the individuals are guided not on initial endowments allocation, available in the beginning of a trajectory, but on their sum with a flow of goods, received from participation in contracts of other coalitions.

(3) The signing of a new barter contract by the members of a coalition with, probably, subsequent breaking of intra-coalitional contract, potentially can also initiate the break of contracts in other coalitions (if the individual participates in contract).

The difficulties which may happen here are caused by item (3), which in general case can entail hardly predictable character of contractual process under hypothesis **(IB)** — instantaneous breaking of the contracts. A problem, with which we encounter, can be illustrated by the following example. Let us consider economy with three agents and let all coalitions are permitted. Let the coalition $\{1, 2, 3\}$ be active and its members sign a new contract, which being realized creates the following situation. All three agents can wish to partially break off the contracts in bilateral coalitions. However the break of the contracts in coalition changes consumption bundles of its members and, thus, influences a measure of desirable break of the contracts in other

[27] Since the law (4.8) for a contract $w = \beta^{min}(x, v)v$ is equivalent to the law (4.11) relative to v.

coalition. For example, in a coalition $\{i, j\}$ the agent i would like to break off intra-coalitional contract in volume $\frac{1}{2}$, if a coalition $\{i, k\}$ does not break anything. However, if the agent k will break the coalitional contract in volume $\frac{1}{3}$, the agent i changes opinion on volume of break of contract in $\{i, j\}$ on the break in a measure $\frac{1}{3}$. However, on the opinion of individual k can affect the break of contract in a coalition $\{k, j\}$, initiated by the agent j, the opinion of which is influenced by a measure of break of contract in $\{i, j\}$, it may be $\frac{1}{2}$ or *e.g.* $\frac{1}{3}$, and etc. So, what will take place as a result of such cyclic breaking of the contracts?

Before the study a general case, it is reasonable to analyze some particular variants of proper-contractual process.

One of opportunities is to try to modify the assumption **(IB)**. Really, one can assume that, similarly to the signing of new contract, a *breaking* of contracts in a coalition occurs only *if* this *coalition is active*. The passive coalition neither able to sign a new contract nor break signed earlier contracts. Further this behavioral hypothesis is designated as **(IBA)** (*i.e.* **(IB)** only for active coalitions).

The second opportunity consists in restriction of a class $\mathbf{K}$ of permitted coalitions. For example, it can be a class all paired (two elements) coalitions. Then, if the coalition $\{i, j\}$ is active in contractual process, it can entail breaks of contracts only in coalitions $\{i, k\}$ and $\{k, j\}$, where $k \neq i, j$. Moreover if *only one coalition* is active at a current moment of time then only it's members, i and j, can initiate the breaking of contracts and the transition is unequivocally determined.

5.1. Breaking of contracts in active coalitions

The serious problems do not arise, if in addition to assume that only *non-intersected* coalitions can be *active* when time is going, of course for different time moments there may be the different sets of active coalitions. Besides we shall assume, that if in some active coalition there is an agent, aspiring to break contracts *without signing of new contract*, then *only the breaking* is realized in contractual process for this coalition at a current moment of time. Further we shall describe this process in details.

Let some coalition-contractual trajectory be given, described by a set of maps $\{v^S(t)\}_{S\in\mathbf{K}}$, adequate to some set of *permitted* coalitions $\mathbf{K} \subset \mathbf{2}^{\mathcal{I}}$. These maps (see (3.3)) are absolute continuous on $[0, \infty)$ and satisfy: $\forall\, t \geq 0$

$$v^S(t) = (v_i^S(t))_{\mathcal{I}} : \sum_{i\in S} v_i^S(t) = 0 \;\&\; v_i^S(t) = 0,\ \forall\, i \notin S.$$

The trajectory is related with these maps by equality

$$x(t) = \omega + \sum_{S\in\mathbf{K}} v^S(t), \quad t \geq 0.$$

Notice, that now, in difference with **IUB**-trajectory by Definition 4.1, the state of trajectory $x(t)$ can already not satisfy to a condition of absence of desire to partially break intra-coalitional gross contracts $v^S(t)$ for the agents from permitted coalitions $S \in \mathbf{K}$, *i.e.*, item **(i)** in coalitional context of Definition 4.1 can be broken. This is the basic difference of **CUB**-process with breaking only in active coalitions.

Let in a moment $t \geq 0$ there is a list of active coalitions $\Upsilon(t) = \{S_1, \dots, S_k\} \subset \mathbf{K}$, which are *pairwise non-intersected*. At the given moment t the exchange processes can go only in coalitions from this list. Thus for each of active coalitions (we shall denote a current one as S) there can be realized one of two opportunities.

First: the members of coalition S do not desire while to break gross coalitional contract $v^S(t)$, and there is an opportunity to sign a new mutually beneficial contract (if not, then coalition is idle, *i.e.*, the zero contract is signed).

Second: there is an individual in S, aspiring to break the coalitional contract $v^S(t)$.

Further we consider specified opportunities consistently.

Let at a moment t the first opportunity is realized and coalition S signs a momentary mutually beneficial contract $w^S = (w_i^S)_S$. Now conditions determining a situation of contracts' breaking are similar to described in Lemma 4.1 and adapted to coalition-contractual trajectory. Really, the favourable situation for the breaking of gross contract $v^S(t)$ as a result of the new contract realization can happen only if for some $i \in S$

$$\langle \nabla u_i(x_i(t)), v_i^S(t) \rangle = 0$$

takes place, this is an analogue of a condition (4.1). Similarly, the breaking of contract $v^S(t) \neq 0$ occurs if and only if for one of agents, satisfying to the previous condition

$$\langle h_i^S(x, v^S(t)), w_i^S \rangle < 0, \quad h_i^S(x, v^S(t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t)) v_i^S(t) \tag{5.1}$$

takes place. Now, if only one agent $i \in S$ aspires to break contracts, for the agents from S the law of change of a trajectory can look like

$$\dot{v}^S(t) = \lambda_i^S(x, v^S, w^S) v^S(t) + w^S, \quad \lambda_i(x, v^S, w^S) = -\frac{\langle h_i^S(x_i, v_i^S(t)), w_i^S \rangle}{\langle h_i^S(x_i, v_i^S(t)), v_i^S(t) \rangle}.$$

If more than one agent of the coalition aspire to break coalitional contract, instead of $\lambda_i^S(x, v^S, w^S)$ in the law of a trajectory one have to take a minimum from values of this type relative to the set of all such agents. For $S \in \Upsilon(t)$ let us determine

$$\lambda_S^{min}(x, v^S(t), w^S) = 0 \bigwedge \min \left\{ \frac{-\langle h_i^S(x, v^S(t)), w_i^S \rangle}{\langle h_i^S(x, v^S(t)), v_i^S(t) \rangle} \;\middle|\; i \in S : \langle \nabla u_i(x_i), v_i^S(t) \rangle = 0 \right\} \tag{5.2}$$

and recall that

$$L_S^c = \{ w \in L \mid w = (w_i)_{\mathcal{I}} : \sum_{i \in S} w_i = 0 \;\&\; w_i = 0, \; \forall\, i \notin S \}$$

denotes the space of the possible contracts for coalition $S \subseteq \mathcal{I}$.

Let for some $i \in S$ the breaking of gross coalitional contract is favourable, *i.e.*

$$\langle \nabla u_i(x_i), v_i^S(t) \rangle < 0$$

takes place. We may think, that in such a case the new contract does not obtain, and there is *only the break* of current contract $v_i^S(t)$. The latter means, that derivative of coalitional contract should be proportional to a vector $-v^S(t)$, that allows to postulate the law[28]

$$\dot{v}^S(t) = -v^S(t).$$

As a result we are coming to the following definition.

Definition 5.1. *A set* $\{v^S(t)\}_{S\in\mathbf{K}}$ *of absolutely continuous maps* $v^S(\cdot) : [0, +\infty) \to L^c_S$ *is called coalitional proper contractual trajectory under hypotheses* **(IBA)**, **(CUB)**, *if the following conditions are satisfied*:

(**i**) *For every* $t \geq 0$ *there is defined a list* $\Upsilon(t) = \{S_1, \ldots, S_k\} \subset \mathbf{K}$ *of active pairwise non-intersected coalitions, in which and only in them, contractual processes may go on*;

(**ii**) *The derivative of a trajectory is defined by a set of momentary mutually beneficial contracts* $\{w^S\}_{S\in\Upsilon(t)}$, *signed among the members of active at the moment* $t \geq 0$ *coalitions* $\Upsilon(t) \subset \mathbf{K}$, *and obeys the law*

$$\dot{v}^S(t) = \lambda^{min}_S(x, v^S(t), w^S)v^S(t) + w^S, \ \ S \in \Upsilon(t), \tag{5.3}$$

where if

$$\langle \nabla u_i(x_i(t)), v^S_i(t)\rangle \geq 0, \quad \forall\, i \in S \in \Upsilon(t)$$

the value $\lambda^{min}_S(x, v^S(t), w^S)$ *is defined by formula* (5.2). *If*

$$\langle \nabla u_i(x_i(t)), v^S_i(t)\rangle < 0, \ \textit{for some} \ \ i \in S \in \Upsilon(t)$$

then $\lambda^{min}_S(x, v^S(t), w^S) = -1$ *and* $w^S = 0$ *takes place.*

(**iii**) *For* $T \in \mathbf{K} \setminus \Upsilon(t)$, i.e., *if coalition* T *at moment* t *is passive (non-involved in contractual process), then* $\dot{v}^T(t) = 0$ *and gross intra-coalitional contract does not change.*

5.2. Proper contractual bilateral CUB-process

In this section we shall assume, that the set $\mathbf{K}$ of permitted coalitions in contractual process is exhausted by all paired coalitions, *i.e.*, we consider the case when

$$\mathbf{K} = \{\{i, j\} \mid i, j \in \mathcal{I} \ \ \& \ \ i \neq j\}.$$

In addition we shall assume, that at any current time moment $t \geq 0$ *only one* from paired coalitions $\{i, j\}$ can sign a new mutually beneficial contract w^{ij}.

Under these hypotheses one can describe proper-contractual process, adequate to postulates **(IB)** and **(CUB)**. Really, as only paired coalitions are permitted then the breaking of contracts initiated by the signing of a new contract may proceed in all coalitions in a predictable way and in conformity with **(IB)** — instantaneous breaking of all unprofitable contracts. Let us describe this process.

Let to a moment $t \geq 0$ for any permitted coalition $S = \{i, j\}$ a state of a trajectory $x(t) = \sum v^{ij}(t)+\omega$, which is defined via a family of gross coalitional contracts $v^{ij}(t)$, obey to a condition of absence of desire to partially break them. This can be expressed in a form

$$\langle \nabla u_k(x_k(t)), v^{ij}_k(t)\rangle \geq 0, \ \ \forall\, k \in \{i, j\}, \ \ \forall\, i, j \in \mathcal{I}.$$

[28] This law of trajectory's change can be deduced analytically, if in a basis of the analysis one takes a trajectory by (4.11), for which specified situation is realized in limit when $\beta^{min} \to 1$.

Fix further coalition $S = \{i, j\}$, which we think to be active one at a moment t, and let its members sign a new contract w^{ij}. Then at moment $\tau = t + \Delta t > t$ the current consumption bundles will look like:

$$x_k(\tau) = \omega_k + \sum_{m \neq i,j} v_k^{km}(t) + v_k^{ki}(t)\alpha_k^i(w^{ij}, \Delta t) + v_k^{kj}(t)\alpha_k^j(w^{ij}, \Delta t), \quad k \neq i, j;$$

$$x_i(\tau) = \omega_i + \sum_{k \neq i,j} v_i^{ik}(t)\alpha_k^i(w^{ij}, \Delta t) + \alpha^{ij}(w^{ij}, \Delta t)[v_i^{ij}(t) + \Delta t w_i^{ij}];$$

$$x_j(\tau) = \omega_j + \sum_{k \neq i,j} v_j^{jk}(t)\alpha_k^j(w^{ij}, \Delta t) + \alpha^{ij}(w^{ij}, \Delta t)[v_j^{ij}(t) + \Delta t w_j^{ij}].$$

Here all values $\alpha_k^i(w^{ij}, \Delta t)$, $\alpha_k^j(w^{ij}, \Delta t)$ and $\alpha^{ij}(w^{ij}, \Delta t)$ are between zero and one and define saved volumes of gross coalitional contracts depending on the new contract w^{ij} and time interval of its realization $\Delta t \approx 0$. Thus for $\Delta t = 0$ all of them are equal to unit. Notice that only individuals i and j can wish to break off the signed earlier contracts. Moreover, only contracts between i and agents from

$$I(i) = \{k \in \mathcal{I} \mid k \neq i, \langle \nabla u_i(x_i(t)), v_i^{ik}(t) \rangle = 0\},$$

can be broken, similarly between j and agents from

$$I(j) = \{k \in \mathcal{I} \mid k \neq j, \langle \nabla u_j(x_j(t)), v_j^{jk}(t) \rangle = 0\}.$$

If simultaneously $j \notin I(i)$ and $i \notin I(j)$, then only the contracts between one of the members of a coalition $\{i, j\}$ and non-members of the coalition can broken off. In this case $\alpha^{ij}(w^{ij}, \Delta t) = 1$ for small Δt. In opposite case it is possible $\alpha^{ij}(w^{ij}, \Delta t) < 1$, *i.e.* coalition $\{i, j\}$ can also be involved in the process of contracts' breaking.

Further, as well as earlier, by virtue of the Taylor's formula

$$\nabla u_i(x_i(t + \Delta t)) = \nabla u_i(x_i(t)) + \nabla^2 u_i(x_i(t))(\Delta t \dot{x}_i(t) + o(\Delta t)) + o(\Delta t \dot{x}_i(t) + o(\Delta t)),$$

and similar representation for j. Besides by virtue of previous considerations we have

$$\dot{x}_i(t) = \sum_{k \neq i,j} v_i^{ik}(t)\lambda_k^i(w^{ij}) + \lambda_i^j(w^{ij})v_i^{ij}(t) + w_i^{ij};$$

where

$$\lambda_k^i(w^{ij}) = \frac{\partial \alpha_k^i(w^{ij}, \Delta t)}{\partial \Delta t}|_{\Delta t = 0}, \quad \lambda_i^j(w^{ij}) = \frac{\partial \alpha^{ij}(w^{ij}, \Delta t)}{\partial \Delta t}|_{\Delta t = 0}.$$

Here values $\lambda_k^i(w^{ij})$ and $\lambda^{ij}(w^{ij})$ one can find solving a system of the equations, received from the first order necessary conditions (now they are also sufficient). Really, if $I(i)^b \subseteq I(i)$ is the set of contractors for i, for which the break of contracts is really occurs, formally for them $\lambda_k^i(w^{ij}) < 0$, then it should be

$$\langle \nabla u_i(x_i(t + \Delta t)), v_i^{ki} \rangle = 0, \ k \in I(i)^b.$$

Using further the representation of a gradient, after necessary transformations, division on Δt, with the subsequent passing to limit in $\Delta t \to 0$, we come to a linear system of the equations for $\lambda_k^i = \lambda_k^i(w^{ij})$:

$$\langle \nabla^2 u_i(x_i(t))\dot{x}_i(t), v_i^{ki} \rangle = 0, \ k \in I(i)^b \quad \Longleftrightarrow$$

$$\sum_{m\in I(i)^b} v_i^{ki}[-\nabla^2 u_i(x_i(t))]v_i^{mi}\lambda_m^i = v_i^{ki}\nabla^2 u_i(x_i(t))w_i^{ij}, \quad k\in I(i)^b. \tag{5.4}$$

Let us show now, that due to assumption (**S**) this system has an unique solution only if the system of vectors $\{v_i^{ki}\}_{k\in I(i)^b}$ is linearly independent. Really, from (**S**) the matrix $A=-\nabla^2 u_i(x_i(t))$ is positive definite and, therefore, it has a square root, *i.e.*, there exists a symmetric positive definite matrix $B=\sqrt{A}$, satisfying $B^2=A$. Now the matrix of coefficients of system is represented as a Grama matrix $||\langle b_k,b_m\rangle||_{k,m\in I(i)^b}$ for the system of vectors $b_k=Bv_i^{ki}$, $k\in I(i)^b$, and system (5.4) can be rewritten in a form

$$\sum_{m\in I(i)^b}\langle b_k,b_m\rangle\lambda_m^i=-\langle b_k,Bw_i^{ij}\rangle, \quad k\in I(i)^b. \tag{5.5}$$

It is known from linear algebra, that Grama matrix for a system of vectors is nonsingular one only when the system is linearly independent. However as soon as B is nonsingular matrix, which determines nonsingular linear transformation, the system of vectors $\{b_k\}_{k\in I(i)^b}$ is linearly independent if and only if the system $\{v_i^{ki}\}_{k\in I(i)^b}$ is linearly independent.

Recall, that in general it is not enough that system (5.4) has a unique solution it is also necessary, that the solution is non-positive in each component.

Actually, in some sense a solution of system (5.4) always exists, because values λ_m^i can be found as an limiting solution of a problem of convex optimization for a given (fixed) $\Delta t>0$ on the compact set of variables $0\le\alpha_m^i\le 1$. However, if the system of vectors $\{v_i^{ki}\}_{k\in I(i)}$ is linearly dependent, this solution may be not unique (in spite of the fact that functions u_i are strictly concave).

It is not easy to take into account all specified difficulties, which may appear in described context of proper contractual process — **CUB**-process with paired coalitions, it is too cumbersome for a first sight. This is why we specify below only the law of change of trajectories in non-degenerated case: it is a situation, when system (5.4) or, accordingly, (5.5), has an unique non-positive solution for both agents, i and j. There are two variants of this situation.

First. Assume, that in a coalition $\{i,j\}$ *no agent aspire* to break coalitional contract $v^{ij}(t)$ after realization of momentary contract w^{ij}. It means, that for i either $\langle\nabla u_i(x_i(t)),v_i^{ij}\rangle>0$, or $\langle\nabla u_i(x_i(t)),v_i^{ij}\rangle=0$ and in the solution of system (5.5) (which is unique!) takes place $\lambda_i^j(w^{ij})=0$. Similar properties should be carried out for the individual j. Thus, in this case $j\notin I(i)^b$ and $i\notin I(j)^b$ simultaneously. Then at the point t the law looks like:

$$\begin{cases} \dot v^{ij}(t) &= w^{ij}, & \\ \dot v^{ik}(t) &= \lambda_k^i(w^{ij})v^{ik}(t), & k\in I(i)^b,\\ \dot v^{jk}(t) &= \lambda_k^j(w^{ij})v^{jk}(t), & k\in I(j)^b,\\ \dot v^{km}(t) &= 0, & \text{for other pairs} \quad (k,m).\end{cases}$$

Here all values $\lambda_k^i(w^{ij})$ and $\lambda_k^j(w^{ij})$ are found as a solution of system (5.5) (for j its own similar).

Second. Let in the coalition $\{i,j\}$ *one* of agents *aspire* to break coalitional contract $v^{ij}(t)$ after realization of momentary contract w^{ij}. It is possible only if, for example for agent i, $\langle\nabla u_i(x_i(t)),v_i^{ij}\rangle=0$ and if in the solution of system (5.5) we have $\lambda_i^j(w^{ij})<0$ (or similar conditions for j). Now define $\lambda_{min}^{ij}=\lambda_i^j(w^{ij})\wedge\lambda_j^i(w^{ij})<0$ and assume that individual i wishes to break off contract $v^{ij}(t)$ in the greatest measure, *i.e.*, $\lambda_i^j(w^{ij})\le\lambda_j^i(w^{ij})$. It is a situation, in which the individual i imposes to the agent j the greater break of gross barter contract, than

j wishes. However it is nothing to do for j, he/she can only accept such rules of game, in so doing changing the initial plans of breaking of the contracts with other individuals. These new volumes of break should be found as a solution of system similar to (5.4), but for the agent j and with fixed unknown variable at $v_j^{ij}(t)$, where one has to take λ_{min}^{ij}. Thus, volumes λ_m^j of breaking of bilateral contracts between j and other agents now can be found from system

$$\sum_{m \in I(j)^b, m \neq i} v_j^{kj}[-\nabla^2 u_j(x_j)] v_j^{mj} \lambda_m^j = v_j^{kj}[\nabla^2 u_j(x_j)](w_j^{ij} + \lambda_{min}^{ij} v_j^{ij}), \; k \in I(j)^b, k \neq i.$$

Of course, everything said above can be correct only if all values to be found are non-positive (otherwise one needs to replace positive values by zero and to continue the search of solution...) As a result, in this case we come to a system of a type

$$\begin{cases} \dot{v}^{ij}(t) &= \lambda_{min}^{ij} v^{ij} + w^{ij}, & \\ \dot{v}^{ik}(t) &= \lambda_k^i(w^{ij}) v^{ik}(t), & k \in I(i)^b, \\ \dot{v}^{jk}(t) &= \lambda_k^j(w^{ij}) v^{jk}(t), & k \neq i, \; k \in I(j)^b, \\ \dot{v}^{km}(t) &= 0, & \text{for other pairs} \quad (k,m). \end{cases}$$

We finish the description of bilateral contractual process at this point — basic character of this should be clear, but there is a wide variety of arising variants, which are rather cumbersome. This essentially reduces prospects on hereinafter effective analysis of convergence in a more-or-less general case.

6. PROPER-CONTRACTUAL PROCESSES: FINAL SPECIFICATIONS AND ASSOCIATED PRICE PROCESS

In the previous sections there were considered basic hypotheses about behavior of the individuals, adequate to contractual processes with partial breaking of the contracts. Besides there were developed some variants of proper contractual processes (trajectories), which corresponds to different combinations of these behavioral hypotheses. The elaborated processes actually were described in the terms of differential inclusions with an autonomous right part, having form

$$\dot{x}(t) \in F(x), \quad x(0) = \omega, \quad t \in [0, +\infty).$$

However for the analysis of convergence of contractual processes this form is not quite convenient, and already for process without breaking of the contracts, for the convergence (to Pareto boundary) there are required some additional assumptions, ensuring the contractual process is going fast enough (see section 2.4 and footnote 16)). We could formulate the necessary hypotheses in general terms of process, however more convenient form is simply to add in the description of process some *trade rule*, which unequivocally determines process as a whole. Doing so, we actually fix some selector of point-to-set mapping, described via differential inclusion. Thus, some additional conditions on a contractual trajectory will be made out as the requirements to a rule of trade. Further we shall consider a rule of trade close to stated in sections 2.3, 2.4, but in an adequate form for contractual processes.

6.1. Trade rule for a contractual trajectory

Let us consider a coalitional proper-contractual trajectory, corresponding to **CUB**-process by Definition 5.1. The trajectory is set by collection $\{v^S(t)\}_{S\in\mathbf{K}}$ absolutely continuous maps $v^S(\cdot) : [0, +\infty) \to L^c_S$, satisfying to conditions **(i)**–**(iii)**. Now we have to specify two things.

First. In item **(i)** it is necessary to postulate the law, revealing for a current moment $t \geq 0$ a set of active coalitions $\Upsilon(t) = \{S_1, \ldots, S_k\} \subset \mathbf{K}$. Apparently, the simplest way to solve this problem is, that for each permitted coalition $S \in \mathbf{K}$ to set open, with *infinite measure*[29] a subset $U_S \subset [0, +\infty)$ of time moments, in which the coalition is active. Further let's define

$$\Upsilon(t) = \{S \in \mathbf{K} \mid t \in U_S\}$$

and also postulate, that for each $t \geq 0$ set $\Upsilon(t)$ (can be empty) consists from pairwise non-intersected coalitions.

Second. In item **(ii)** it is necessary to specify *which* of the mutually beneficial contracts is signed by the members of an active coalition. And now occurs a map, that it is possible to name a *trade rule*. As soon as the fact that a contract is mutually beneficial for the members of a coalition depends only on current consumption bundles we can assume that for each coalition $S \in \mathbf{K}$ there is determined a *continuous* map

$$w^S : \mathcal{A}(X) \to L^c_S = \{w \in L \mid w = (w_i)_{\mathcal{I}} : \sum_{i\in S} w_i = 0 \ \&\ w_i = 0,\ \forall\, i \notin S\},$$

such that

$$u_i(x_i + w^S_i(x)) > u_i(x_i) \quad \Longleftrightarrow \quad \exists\, \nu \in L^c_S : u_i(x_i + \nu_i) > u_i(x_i)\ \forall\, i \in S, \tag{6.1}$$

and

$$\nexists\, \nu \in L^c_S : u_i(x_i + \nu_i) > u_i(x_i)\ \forall\, i \in S \quad \Rightarrow \quad w^S(x) = 0.$$

The value $w^S(x) \in L^c_S$ unequivocally specifies the contract, which will be signed by a coalition S at moment t under two conditions: 1) if $S \in \Upsilon(t)$, *i.e.*, the coalition has to be active, and 2) if $x(t) = x$, *i.e.*, if allocation achieved by a trajectory at moment t coincides with x. The stated requirements to the map $w^S(\cdot)$, $S \in \mathbf{K}$ mean, that at each time moment when the coalition is active some mutually beneficial contract is signed, if there is such opportunity at all; if no, the contract does not signed (more precisely, the zero contract is signed). Notice that when utilities are differentiable instead of (6.1) it is appropriate to apply condition presented for gradients (derivatives by direction):

$$\langle \nabla u_i(x_i), w^S_i(x)\rangle > 0 \quad \Longleftrightarrow \quad \exists\, \nu \in L^c_S : u_i(x_i + \nu_i) > u_i(x_i)\ \forall\, i \in S.$$

Really, requirement (6.1) seems quite correct for discrete time presentation, however when time is continuous $w^S(x(t))$ is a momentary contract which defines not only beneficial exchange proportions but also the speed of exchange processes (multiplying $w^S(x(t))$ by a number over unit we increase the speed of exchange and for positive but less than unit numbers a speed is decreasing and etc.). Integrating $w^S(x(t))$ over time on interval $[t, t + \Delta t]$ one can find a gross contract, obeying (6.1) in the moment t and when a time of its realization is $\Delta t > 0$ (see comments from section 3.2).

[29] This property is necessary so that the coalition was capable to realize the interests at least in infinity.

So, up to the moment *coalitional proper-contractual process* and the trajectory are completely described, for it is enough in Definition 5.1 to add the additional factors: sets U_S, $S \in \mathbf{K}$, determining current structure of active coalitions $\Upsilon(t)$, and map $w^S(\cdot)$, $S \in \mathbf{K}$, specifying contracts signed by active coalitions $w^S(x(t))$, which is necessary to use in the law (5.3).

Actually we have described not only coalitional, but also *aggregated* proper-contractual trajectory, determined in Definition 4.1. Really, aggregated trajectory turns out from coalitional one if $\mathbf{K} = \{\mathcal{I}\}$. Though, certainly, in this case only a trade rule, formalized by map $w^{\mathcal{I}}(\cdot)$ will be actually applied (requirements to this map can be relaxed a little bit).

In closing of this section we wold like to do an important remark. The *right hand side* of equation determining the law of contractual trajectory (4.8), (5.3), is *discontinuous* in general — in spite of it is defined unambiguously and is formulated via continuous functions! Of course this problem (perhaps imperceptible for a first view) is arisen because of parameter $\lambda^{min}(x, v)$ defined by formula (4.7) (or analogous $\lambda_S^{min}(x, v^S, w^S)$ by (5.2)) which vanishes in an open area $x \in \mathcal{A}(X)$: $\langle \nabla u_i(x_i), x_i - \omega_i \rangle > 0$ for all i but in general it is non-zero on its boundary, *i.e.*, at the points $(x_1, x_2, \ldots, x_n) \in \mathcal{A}(X)$ where $\langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0$ at least for one i. Thus the law of proper contractual trajectory is described via differential equation with a discontinuous right hand part. Moreover in such a case already the concept of solution requires an accurate definition. Solution is a *continuous* function of time, which satisfies the law of trajectory change for *almost all* time moments. The solution of this kind defines a contractual trajectory in an appropriate way.

Notice that classical theorems on existence, uniqueness and continuous dependence over initial data cannot be applied to equations with discontinuous right hand parts since their right parts do not obey Lipshitz condition. However by now there is a theory of equations of this type in which appropriate theorems (on existence, uniqueness and continuous dependence) have proven and these theorems are applicable to equations describing contractual processes, see Filippov (1985).

6.2. Price process, associated with a contractual trajectory

The purpose of this paragraph is to discuss available opportunities to connect with a contractual trajectory some parallel dual price process. This idea seems to be rather tempting, since being successfully realized, it would allow us, on the one hand, better to reveal interrelations and peculiarities of contractual trajectories with processes known in the literature (first of all with tâtonnement) and, on the other hand, it is impossible to wave away from the fact, that in real economy the barter processes are going mainly with use of money, which are exchanged on goods and services in proportions given by prices. A quest about an appropriate price dynamics is also important for better understanding of how markets function. However the realization (successful) of this idea encounters with some difficulties. A possible approach to realize the idea will be discussed below.

The most natural way to define current prices $p(t)$ is to take the prices as a vector, specifying exchange proportions in the current barter contract. It would be not bad, if economy has only two goods and, if in the bargain only two individuals have been participated. Really, then if first agent receives from the contract a vector (v^1, v^2), where $v^1 > 0$ and $v^2 < 0$, it means that 2-nd good exchanges on 1-st in proportion $-v^1/v^2$, *i.e.*, (normalized) vector of prices can be chosen as $p = (1, -\frac{v^1}{v^2})$. This vector is possible to determine via an equivalent way, from equations:

$$\langle (v^1, v^2), (p_1, p_2) \rangle = 0, \quad p_1 = 1.$$

This property, equality to zero of scalar product of a vector of the prices on vectors of commodities flows v_i, received by the individuals from contract $v = (v_i)_{\mathcal{I}}$, together with normalization, is the crucial requirement for determination of a price vector in this approach. However, in a general case it is possible to find vector of the prices unequivocally, only if the rank of system of vectors $\{v_i\}_{\mathcal{I}}$ given by contract v is equal $l - 1$ (l is the number of goods), *i.e.*, if $\operatorname{rank}(\{v_i\}_{\mathcal{I}}) = l - 1$. On the other hand, other contracts may coexist at current time moment, and, moreover contracts signed in the past and existing now should also participate in "determination of proportions of an exchange". As a result, for aggregated proper-contractual trajectory the given approach yields the system of equations:

$$\langle p(t), x_i(t) - \omega_i \rangle = 0, \quad \forall\, i \in \mathcal{I}, \quad t \geq 0. \tag{6.2}$$

At the same time, for a coalition trajectory the similar system takes a form:

$$\langle p(t), v_i^S(t) \rangle = 0, \quad \forall\, i \in S,\ \forall\, S \in \mathbf{K}, \quad t \geq 0. \tag{6.3}$$

Moreover, in both cases it should be carried out some normalizing condition, for example, $p_1(t) = 1$.

Clearly, that it is absolutely not certain, that the specified systems has an unique solution: the system can be unsolvable or, contrary, it may have infinity many solutions. However in such case it is possible to try as the prices to take some approximating solution. Certainly, it is necessary to do in some regular way, so that a trajectory $p(t)$, $t \geq 0$ has "not bad" mathematical properties. Having this in mind one can use known methods of a finding of approximating solution, for example, the method of least squares. In so doing so-called "generalized inverse matrix" is entered into consideration, which exists and is unique for any rectangular matrix. Let us pay attention to this method.

Let us consider a system of the linear equations with some rectangular matrix:

$$Ax = b.$$

An approximating solution of system can be found with the help of *generalized inverse* matrix A^+ as $x = A^+ b$, and, if system is overdetermined (a number of independent equations is more than unknown variables), this gives a solution by a method of the least squares. Then, if a rank of matrix is equal to the number of columns, $(A^t A)^{-1}$ exists (this is Grama matrix) and matrix A^+ can be fond as

$$A^+ = (A^t A)^{-1} A^t.$$

In a general case the matrix A^+ is set via the following four conditions:

$$AA^+A = A; \quad A^+AA^+ = A^+; \quad A^+A \ \ \& \ \ AA^+ \ \ \text{symmetrical}.$$

These conditions completely determine a matrix A^+ and were offered by Penrose (Penrose, 1955), which has also proved existence and uniqueness of a matrix A^+. This problem was considered also in Moore's works and papers of other authors, this is why one can meet in the literature the name "Moore-Penrose inverse" (see Greene, 1993 and Searle, Hausman, 1970).

In a general case the matrix A^+ also has formula representation (see Greene, 1993, p. 45), that in particular allows to conclude continuously-differential character of the solution.

So, if to apply a generalized inverse matrix associated price process $p(t)$ to an aggregated trajectory can be determined as

$$p(t) = A^+ e_1, \quad e_1 = (1, 0, \ldots, 0),$$

where the matrix A of dimension $(n+1) \times l$, with rows a_k, $k = 1, \ldots, n+1$ which are equal: $a_1 = e_1$ and $a_k = x_{k-1}(t) - \omega_{k-1}$, $k \geq 2$, accordingly.

For a coalitional trajectory associated price process $p(t)$ can be set by the same formula, in which, however, the matrix A has dimension $(\sum_{S \in \mathbf{K}} |S| + 1) \times l$, first row $a_1 = e_1$, and all other are indexed by pairs (i, S), $i \in S \in \mathbf{K}$ and coincide with vectors v_i^S.

For a proper-contractual trajectory it is also possible to try to determine price process as the differential equation.

Really, for an aggregated trajectory, formal differentiation by time system (6.2) (notwithstanding that it may have not the exact solution!), with the account (4.8), (6.2) yields

$$\langle \dot{p}(t), x_i(t) - \omega_i \rangle + p(t)\dot{x}_i = 0 \;\Rightarrow\; \langle \dot{p}(t), x_i(t) - \omega_i \rangle = -p(t)[\lambda^{min}(x, v)(x_i(t) - \omega_i) + v_i] \;\Rightarrow$$

$$\langle \dot{p}(t), \omega_i - x_i(t) \rangle = p(t) v_i(x), \quad \forall\, i \in \mathcal{I}, \quad \dot{p}_1(t) = 0. \tag{6.4}$$

Further, applying a generalized inverse matrix the system (6.4) can be approximately solved relative to $\dot{p}(t)$ and can be written in an explicit form. It is possible also to try to solve systems (6.2), (6.4) jointly and find a direct dependence of $\dot{p}(t)$ from current consumptions $x(t)$ and an exogenous trade rule $v(x)$.

The similar actions can be made for a coalitional trajectory. Really, differentiating system (6.3), by virtue of Definition 5.1, items **(ii)**, **(iii)**, and applying (6.3), in the case when a new contract w^S is signed by a coalition $S \in \Upsilon(t)$ we find

$$\langle \dot{p}(t), v_i^S \rangle + p(t)\dot{v}_i^S = 0 \quad \Rightarrow \quad \langle \dot{p}(t), v_i^S \rangle = -p(t)[\lambda_S^{min}(x, v^S, w^S) v_i^S + w_i^S] \quad \Rightarrow$$

$$\langle \dot{p}(t), v_i^S(t) \rangle = -p(t) w_i^S(x), \quad \forall\, i \in S, \quad \dot{p}_1(t) = 0. \tag{6.5}$$

In all other cases (the breaking of contract or when a coalition is passive) we obtain

$$\langle \dot{p}(t), v_i^S(t) \rangle = 0, \quad \forall\, i \in S.$$

What preliminary conclusions and observations can be done about suggested variant of price process?

1) The current prices are determined as an average (approximate) vector of exchange proportions over all (gross) contracts, signed and saved (*i.e.* unbroken) during contractual process up to the current time moment. It seems that it corresponds to economic intuition on how, in mass manner, the individuals can find the prices acting in the real markets. For example, similar method is used for a finding of market price for the objects of real estate.

2) We need the analysis of price process to be continued at least for some particular model examples. Moreover, it seems us that an opportunity to apply price process for a finding of suitable Lyapunov's function has to be analyzed, to achieve the basic purpose of the project and to establish convergence of proper-contractual process, at least in limited frameworks.

3) Now it is not quite not clear, how suggested price process is connected with Walrasian tâtonnement. However it is possible to show, that in economy with two agents and two goods the associated price process is practically equivalent to Walrasian tâtonnement. Moreover, if the trajectory converges to equilibrium allocation then it is simple to prove that price process converges to the equilibrium prices.

4) It is possible to consider modifications of described price process, even essentially different price processes, with the purpose to find variant interrelated with the different points of view. For example, it seems tempting to study still non-described idea (now it is not elaborated yet): to put into correspondence to contractual process some optimal control problem, which has a contractual trajectory as an optimal solution. Then one can try to find price process from necessary conditions of optimality.

7. CONTRACTUAL PROCESS IN 2×2 ECONOMY

Before the starting to describe obtained results we consider two particular examples of economy with two individuals and two commodities. These examples are interesting because they reveal in Edgeworth box the geometrical course of contractual processes with partial breaking of contracts. One can easily observe that our process is convergent in these cases.

Further an agent is called *active* at a current time moment t if he/she realizes an breaking of aggregated contract at this moment of time.

7.1. Two examples

Now we consider two particular examples of economy with two individuals and two commodities. These examples are interesting because they reveal in Edgeworth box the geometrical course of contractual processes with partial breaking of contracts. One can easily observe that our process is convergent in these cases.

For both examples positive orthant in 2-dimensional plane presents individual consumption sets, *i.e.*, $X_i = \mathbb{R}^2_+$, $i = 1, 2$. The examples are differentiated via agents' utilities and endowments.

Example 7.1 (Cobb–Douglas utilities). Let preferences be presented by Cobb-Douglas utilities in logarithmic form as follows

$$u_1(x_1, x_2) = \frac{1}{4}\ln x_1 + \frac{3}{4}\ln x_2, \qquad u_2(y_1, y_2) = \frac{3}{4}\ln y_1 + \frac{1}{4}\ln y_2.$$

Consider also the following initial endowments:

$$\omega = (\omega_1, \omega_2) = ((\frac{9}{10}, \frac{1}{10}), (\frac{1}{10}, \frac{9}{10})), \quad \bar{\omega} = \omega_1 + \omega_2 = (1, 1).$$

Then indifference curves for first and second individual going across initial endowments point ω_1 in 1st agent's coordinate system are described by equations:

$$x_2 = \frac{1}{10}\left(\frac{9}{10x_1}\right)^{\frac{1}{3}}, \; x_2 = 1 - \frac{9}{10}\left(\frac{1}{10(1-x_1)}\right)^3.$$

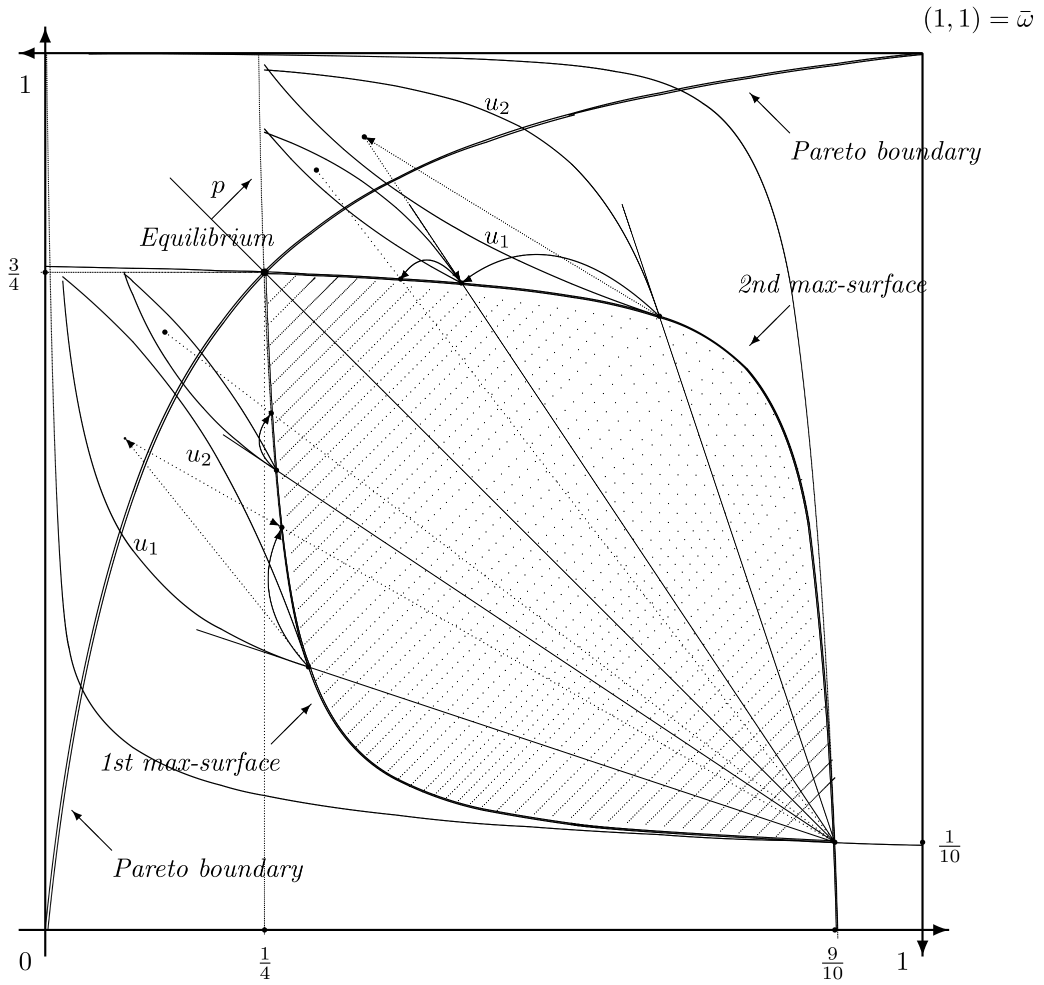

Fig. 2. Contractual process in economy with 2 agents and Cobb–Douglas utilities

Calculations show that Pareto boundary is a curve determined by equation

$$x_2 = \frac{9x_1}{1+8x_1}, \quad 0 \le x_1 \le 1.$$

Finally, a maximal surface is composed via two curves and it is the low envelope for them:

$$x_2 = \left(\frac{3x_1}{40x_1-9}\right), \; x_1 > \frac{9}{40} \quad \& \quad x_2 = \left(\frac{28-31x_1}{37-40x_1}\right), \; x_1 < \frac{28}{31}.$$

An illustration of this example in Edgeworth box is given in Fig. 2.

In considered case proper contractual process is convergent to unique equilibrium $((\frac{1}{4},\frac{3}{4}),(\frac{3}{4},\frac{1}{4}))$. This is developed in the following way: if its trajectory is in limits of maximal surface (shaded area of Fig. 2), then individuals are cooperated and signed some barter contracts giving a rise of utilities. A current consumption point is moving this way as long as it stars to leave the maximal surface limits. If a new contract starts to lead the point behind maximal surface and new position is under control of 1st agent (we call him 'active' in contractual process), this is the left low part of box restricted by budget line, then this agent partially breaks aggregated contract and a current point of trajectory is projected onto maximal surface along to straight line going at initial endowments point. Analogous thing takes place for second agent if a new contract leads the point to the area under 2nd agent control (right upper part of box behind the budget line). As it is shown in figure in both cases next point of trajectory moves over maximal surface and approaches to equilibrium. Thus in the limit our trajectory achieves equilibrium allocation where 1st agent consumption is $(\frac{1}{4}, \frac{3}{4})$.

The following example is well known in literature and presents an economy with multiplicity of equilibria.

Example 7.2 (Exponential utilities). Let preferences be determined by the following utilities functions:

$$u_1(x_1, x_2) = x_1 - 100e^{\frac{-x_2}{10}}, \qquad u_2(y_1, y_2) = y_2 - 110e^{\frac{-y_1}{10}}.$$

Let $\omega = (\omega_1, \omega_2) = ((40, 0), (0, 50))$ be an initial endowments allocation. Now the indifference curves of both agents going across endowments are defined by equations

$$x_2 = -10 \ln \left(\frac{x_1 + 60}{100} \right), \; x_2 = 110 - 110e^{\frac{x_1 - 40}{10}}.$$

Pareto boundary is a straight line defined as

$$x_2 = x_1 - 40 + 10 \ln 110.$$

Maximal surface is composed by means of two curves, relative to 1st and 2nd agents:

$$x_1 = 40 - 10x_2 e^{\frac{-x_2}{10}}, \; x_2 = 11(40 - x_1)e^{\frac{x_1 - 40}{10}}.$$

There are three equilibrium allocations, A, B, C, where 1st agent consumption bundles are the following:

$$x^{(1)} \approx (3.2212, 10.226), \quad x^{(2)} \approx (13.17211, 20.1769), \quad x^{(3)} \approx (32.2579, 39.2627).$$

An illustration of this example in Edgeworth box is given in Fig. 3. Contractual process is developed similar to described above: in the limits of maximal surface agents sign mutually beneficial contracts as long as an allocation behind area restricted by maximal surface is reached. On the other hand then a trajectory tends to leave maximal surface area and a current point is under control of 1st agent (he is active, this touch-dotted line in figure), the agent partially breaks aggregated contract so that the point is projected onto maximal surface along a straight line going across current point and initial endowments one. The similar thing takes place for the points under 2nd agent control. One can see in figure that the next value of trajectory is moved over maximal surface to be closely to one of equilibrium: in the area of 1st agent activity to **C**, in the area of 2nd agent to **A**. Finally, if trajectory "attempts to leave" the limits of maximal surface and is placed exactly on the straight line linked **B** and initial endowments then trajectory is finished at the point **B**.

So we see that in this example the multiplicity of equilibria does not impede contractual process to be convergent. Moreover one can correctly speak about locally stable (**A**, **C**) and unstable (**B**) equilibria relative to contractual processes.

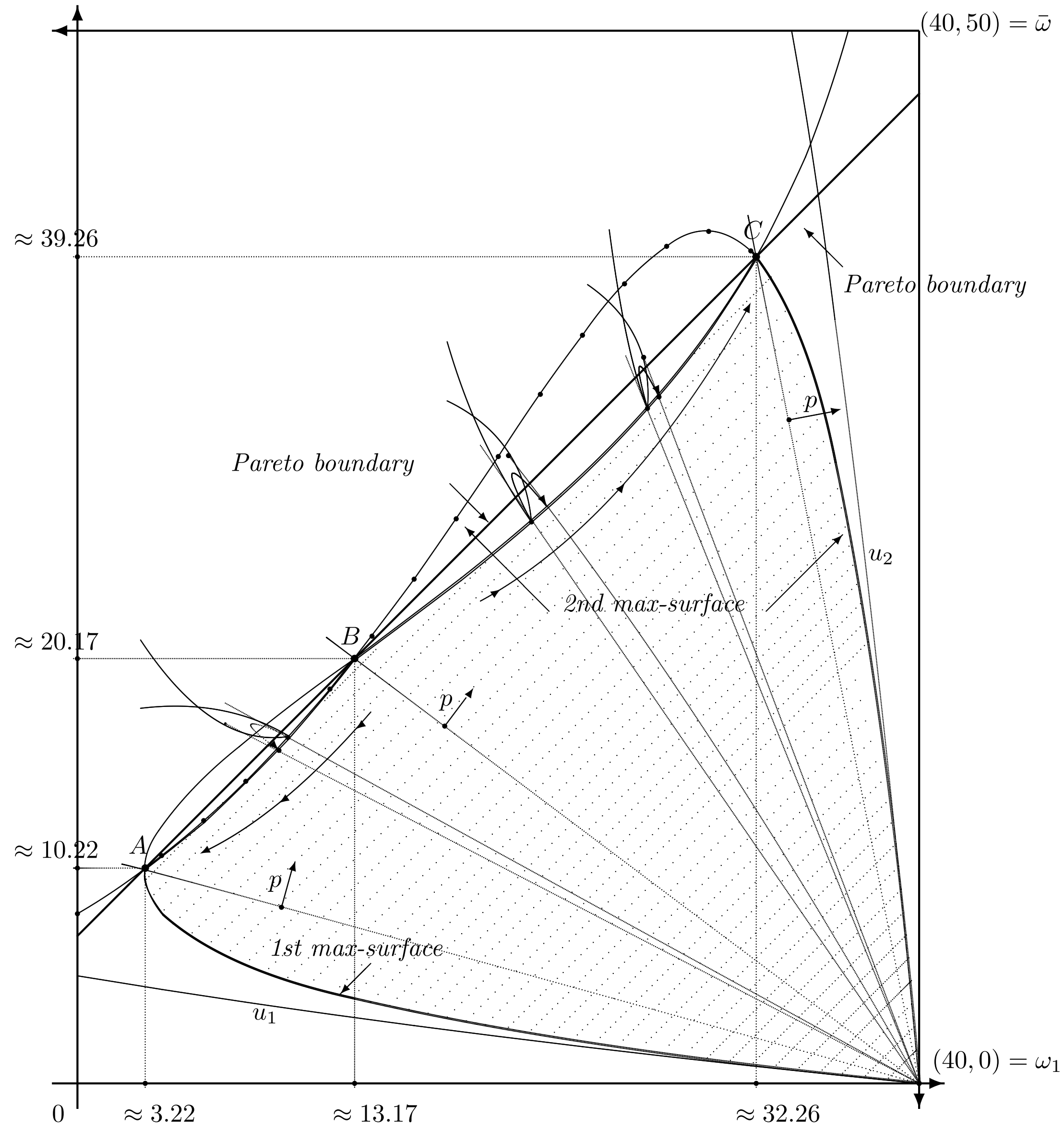

Fig. 3. Contractual process in economy with 2 agents and Exponential utilities

7.2. Analysis of contractual processes in 2×2 economies

The geometry of contractual trajectories in the considered examples says us that at least in economy with two individuals and two goods the contractual process has to converge to equilibrium under rather general assumptions. Really, in addition to imposed above assumption **(S)** that model is smooth we only need to require $\mathcal{P}_i(\omega_i) \subset \mathrm{int}X_i$, $i = 1, 2$, that together with **(S)** actually provides the coincidence of proper contractual allocations with equilibrium ones.

To prove this hypothesis, let us consider some contractual trajectory $x(t) = (x_1(t), x_2(t))$, $t \in [0, +\infty)$ satisfying to Definition 4.1 and defined via some rule of trade

$$v^{\mathcal{I}} : \mathcal{A}(X) \to L^c = \{(v_1, v_2) \in (\mathbb{R}^2)^{\mathcal{I}} \mid v_1 + v_2 = 0\}.$$

As soon as economy has only two agents and $\mathcal{I} = \{1, 2\}$ a sole coalition in which an exchange of commodities may be realized, then trading rule consists of unique map $v^{\mathcal{I}}(\cdot)$, where upper index $\mathcal{I}$ can be omitted. By definition $v_2(x) = -v_1(x)$, $\forall\, x \in \mathcal{A}(X)$ and it is enough to set only function $v_1 : \mathbb{R}^2_+ \to \mathbb{R}^2$, where the vector $(v_1(x_1), -v_1(x_1)) = v^{\mathcal{I}}(x_1)$ is associated with the instant contract, which is signed the members of coalition $\{1, 2\}$ at the moment $t \in [0, +\infty)$ provided that a current allocation is $x(t) = (x_1, x_2)$, $x_2 = \omega_1 + \omega_2 - x_1$. In addition, this function should be continuous and to define mutually beneficial contract $v = v^{\mathcal{I}}(x_1)$: $v \neq 0$ if and only if

$$\exists\, \nu \in L^c : u_i(x_i + \nu_i) > u_i(x_i),\ i = 1, 2 \quad \& \quad \partial_{v_i} u_i(x_i) > 0,\ \ i = 1, 2.$$

In other words, if *there is* at least one *opportunity* for a mutually beneficial exchange, then *one of variants* of such exchange *should be realized* in the form of the mutually beneficial contract; the contract can be *zero* only if *there are* no *opportunities* for a mutually beneficial exchange of commodities.

For contractual process in economy with two individuals only two following alternatives can be realized.

(i) There is an individual such that since some moment τ, almost everywhere on $[\tau, +\infty)$, *only he/she* can be active (probably both are passive); thus utility of this individual monotonously does not decrease along a trajectory, *i.e.*, for example for the first agent, it has to be

$$u_1(x_1(t') \geq u_1(x_1(t), \quad \forall\, t' \geq t \geq \tau.$$

(ii) The case described in **(i)** is not true, i.e, there are monotonously increasing sequences $t'_{k+1} > t'_k$, $t''_{k+1} > t''_k$, $k \in \mathbb{N}$, $t'_k \to +\infty$, $t''_k \to +\infty$ when $k \to +\infty$, such that at the moments t'_k the 1st individual is active, and t''_k are the moments of 2nd agent activity, $k = 1, 2, \ldots$ Thus, utilities of both individuals can oscillate, growing and decreasing, and this situation does not change when time elapses.

Analysis of alternatives **(i)**, **(ii)** is realized in two subsequent lemmas. The first establishes that if alternative **(i)** is true then every limit point of a contractual trajectory is Pareto optimal. The second alternative causes the greatest difficulties and second lemma states that in some sense there is the monotonicity of utilities along a trajectory but it has “piecewise” character.

Lemma 7.1. *Let alternative* **(i)** *be fulfilled. Then each limit point of a contractual trajectory is Pareto optimal. Hence, every* ***interior*** *limit point is equilibrium one.*

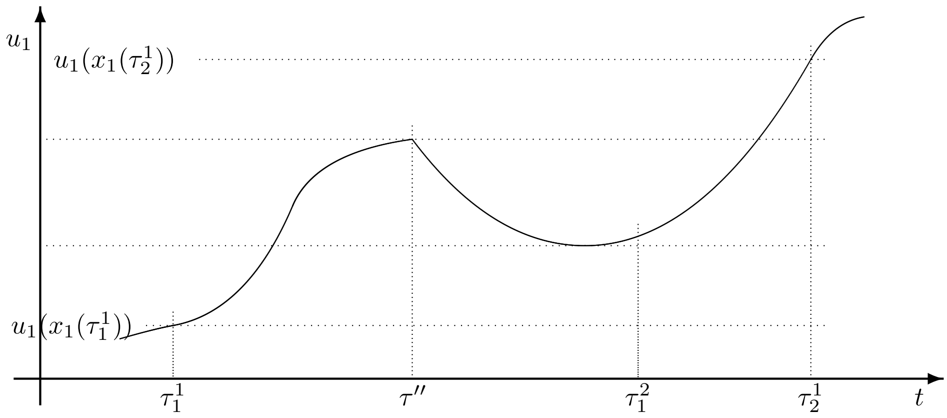

Fig. 4. Utility dynamics of contractual process in 2×2 economy, alternative (**ii**)

The proof of Lemma 7.1 is placed in Appendices.

Remark 7.1. One can easy to see from the proof that this lemma is also true for economy with any number of agents and commodities. It is important only, that alternative (**i**) is fulfilled. Moreover, if economy has only two agents and alternative (**i**) is true then it is easy to prove that all limit points have equal utilities for both agents, that for strictly concave functions is possible only if allocations are equal (see the first part of Theorem 7.1 proof).

Remark 7.2. From the proof of Lemma 7.1 and due to Remark 7.1 one can also conclude convergence of contractual process without break of contracts. However now it will be convergence to some Pareto optimal allocation: it is obvious, that the limit point is not obliged to be equilibrium since it is not a limit point of allocations which are stable relative to partial break of gross contract. However when utilities are strictly concave there is only one limit point, that easily follows from the fact of coincidence of utility (follows from monotonicity along a trajectory) for all individuals in all limit points.

Lemma 7.2. *Let alternative* (**ii**) *be fulfilled. Then there are exist two monotonously increasing sequences of the moments of time* τ_k^i*, for* $i = 1, 2$*, such that* $\tau_k^1 < \tau_k^2 < \tau_{k+1}^1$*,* $\langle \nabla u_i(x_i(\tau_k^i)), x_i(\tau_k^i) - \omega_i \rangle = 0$*,* $i = 1, 2$*,* $\forall\, k \in \mathbb{N}$ *and*

$$u_i(x_i(\tau_k^i)) < u_i(x_i(\tau)) < u_i(x_i(\tau_{k+1}^i)), \quad \forall\, \tau \in (\tau_k^i, \tau_{k+1}^i), \quad i = 1, 2 \tag{7.1}$$

holds.

The proof of Lemma 7.2 is placed in Appendices. The contents of this lemma is illustrated in Figures 4, 5. Fig. 4 demonstrates dynamics of utility for sequences of time moments τ_k^i, $i = 1, 2$, $k = 1, 2, \ldots$ constructed in the proof and described in Lemma 7.2. Possible and impossible dynamics of proper-contractual trajectories are demonstrated in Fig. 5; there is shown, that the only fragment of trajectory, represented in the top part of figure is possible — all other variants result to an impossible cycle contradicting with the choice of the moments τ_k^i.

The main result of the section is the following theorem on convergence of contractual trajectories to equilibrium in economy with two agents and two goods.

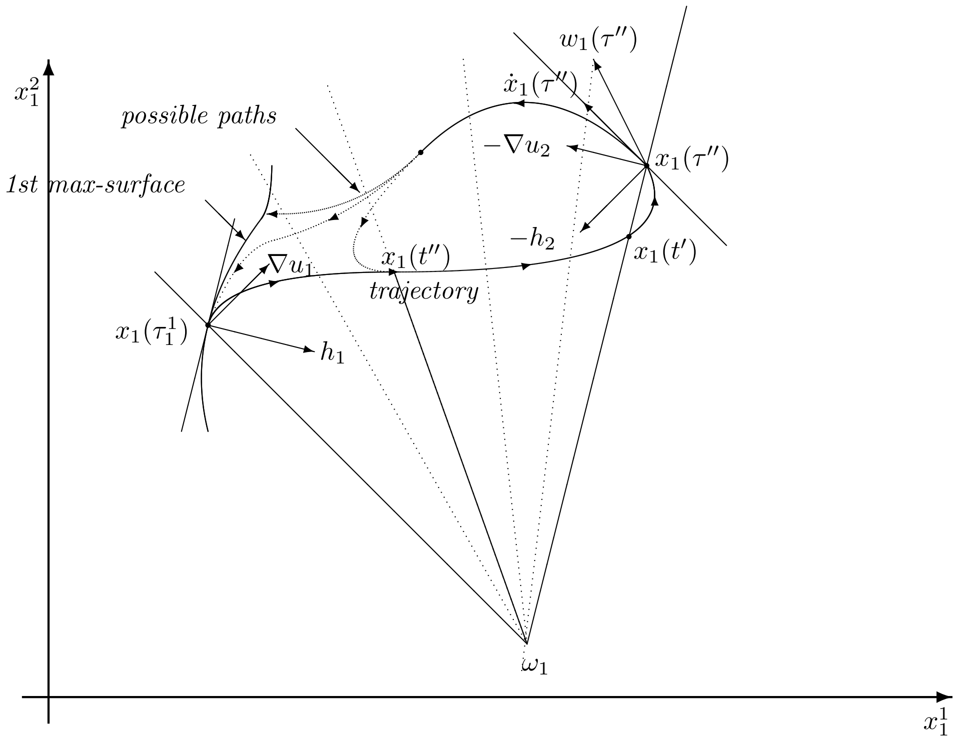

Fig. 5. Possible and "impossible" moving of contractual trajectory in 2×2 economy, alternative (**ii**)

Theorem 7.1. *Let economy have two agents and two commodities. Let utilities be smooth, strictly concave and non-satiated on* $\mathbb{R}_+^2$*. Then for any continuous trading rule the contractual trajectory by Definition 4.1 converges to some proper-contractual allocation. Hence, in conditions when equilibrium allocations coincide with proper-contractual ones, every proper-contractual trajectory converges to equilibrium.*

Proof of Theorem 7.1. In conditions of the theorem the considered above alternatives (**i**), (**ii**) take place. So, it is enough to show, that for any alternative the contractual process converges to proper-contractual allocation.

Let alternative (**i**) be true. Let's prove, that $x_1(t)$ converges to $\tilde{x}_1$ when $t \to +\infty$. To do it let's assume, that the trajectory has two different limit points $\tilde{x}^1 = (\tilde{x}_1^1, \tilde{x}_2^1)$, $\tilde{x}^2 = (\tilde{x}_1^2, \tilde{x}_2^2)$ and $\tilde{x}^1 \neq \tilde{x}^2$. Due to Lemma 7.1 both of them are Pareto optimal and thus $u_1(\tilde{x}_1^1) = u_1(\tilde{x}_1^2)$. Let's assume, for example, that $u_2(\tilde{x}_2^1) \leq u_2(\tilde{x}_2^2)$. Further consider any allocation represented as a convex combination of the given limit points with strictly positive coefficients, for example one can take $\tilde{x}' = \frac{1}{2}\tilde{x}^1 + \frac{1}{2}\tilde{x}^2$. Now, by virtue of strict concavity of utility functions conclude

$$u(\tilde{x}') \gg u(\tilde{x}^1) \quad \Longleftrightarrow \quad u_1(\tilde{x}_1') > u_1(\tilde{x}_1^1),\ u_2(\tilde{x}_2') > u_2(\tilde{x}_2^1),$$

that contradicts Pareto optimality of allocation $\tilde{x}^1$. Thus all limit points of a trajectory coincide and, hence, the trajectory converges.

Further we analyze alternative (**ii**). With this purpose one can apply Lemma 7.2 and consider limit points of sequences $\{x_1(\tau_k^1)\}_{k\in\mathbb{N}}$ and $\{x_1(\tau_k^2)\}_{k\in\mathbb{N}}$. Without lost of generality one can think that these sequences converge itself. Determine

$$\tilde{x}_1^1 = \lim_{k\to\infty} x_1(\tau_k^1), \quad \tilde{x}_2^1 = \omega_1 + \omega_2 - \tilde{x}_1^1, \qquad \tilde{x}_1^2 = \lim_{k\to\infty} x_1(\tau_k^2), \quad \tilde{x}_2^2 = \omega_1 + \omega_2 - \tilde{x}_1^2.$$

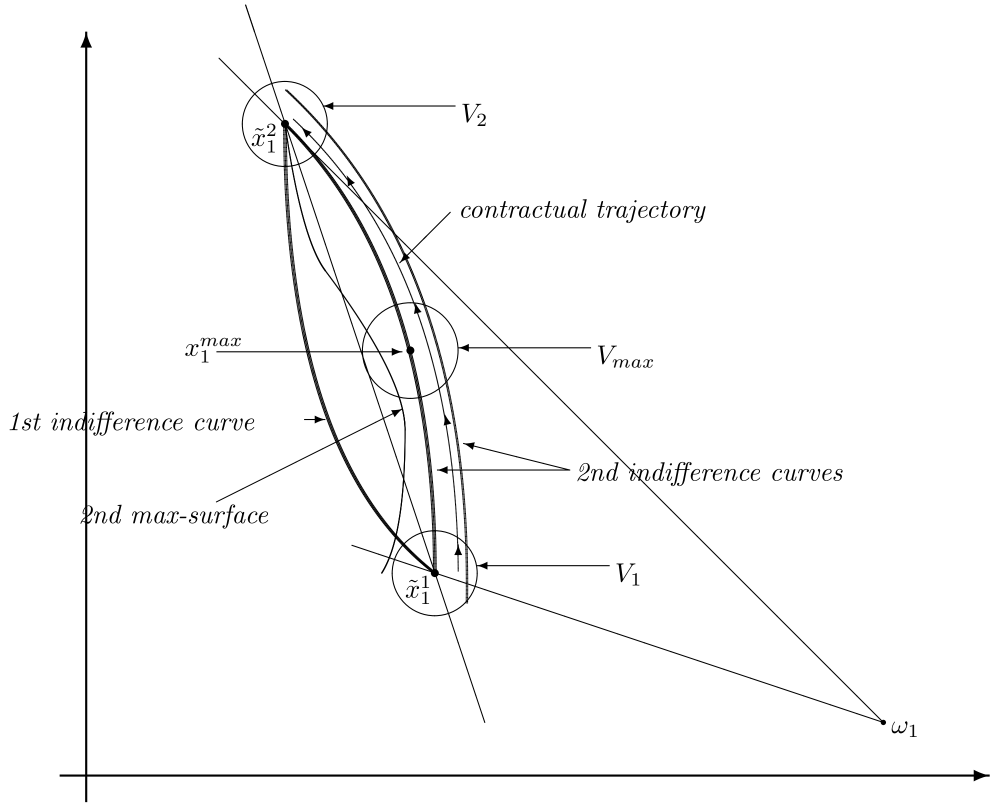

Fig. 6. "The impossible" moving of contractual trajectory along a limit path under alternative (**ii**) and $\tilde{x}_1^1 \neq \tilde{x}_1^2$

In view of (7.1) we have

$$u_1(\tilde{x}_1^1) = \sup_{t \geq \tau_2^1} u_1(x_1(t)) = u_1(\tilde{x}_1^2), \quad u_2(\tilde{x}_2^1) = \sup_{t \geq \tau_2^1} u_2(x_2(t)) = u_2(\tilde{x}_2^2).$$

It is clear, that point $\tilde{x}_1^1$ is on the maximal surface of 1st agent and $\tilde{x}_2^2$ is on the maximal surface of 2nd, and that for both individuals the allocations are equivalent by utility. Further we shall show, that actually coincide not only utilities, but also allocations, *i.e.* $\tilde{x}_1^1 = \tilde{x}_1^2$. For two-goods economy this allocation will be obviously proper-contractual (equilibrium) since it is on the maximal surface of every agent.[30]

Let's assume now that $\tilde{x}_1^1 \neq \tilde{x}_1^2$. These points are on a common indifference curve of 1st agent, and, accordingly, the points $\tilde{x}_2^1 \neq \tilde{x}_2^2$ are on an indifference curve of 2nd individual. Reasoning in Edgeworth box, for example in coordinate system of 1st agent, we see that two points $\tilde{x}_1^1 \neq \tilde{x}_1^2$ are connected by two continuous curves which are the pieces of boundary of two (convex) sets of a utility level. Let's connect the specified points by a linear segment, *i.e.*, consider the set $\{\gamma\tilde{x}_1^1 + (1-\gamma)\tilde{x}_1^2 \mid 0 < \gamma < 1\}$. By virtue of strict concavity of utility functions, the value of utility at points of this segment is strictly more than utility level at its ends for both individuals. This implies that for one of agents the part of indifference curve, going through the points $\tilde{x}_1^1$, $\tilde{x}_1^2$ and placed strictly between these points, cannot intersect maximal surface of the agent, see Fig. 6 (every ray going from initial endowments through one of considered points first intersects with one of two indifference curves and then it hits at a point of segment; therefore

[30] It is not sufficient in general, but it will be so if the allocation is Pareto optimal.

(via concavity) when point moves along the ray utility increases at the point of intersection with indifference curve). Let this be a case of 2nd agent indifference curve. Further for the 2nd agent indifference curve let us find a point x_1^{max} where 1st agent utility is maximal, *i.e.*, define x_1^{max} from relation

$$u_1(x_1^{max}) = \max\{u_1(x_1) \mid u_2(\omega_1 + \omega_2 - x_1) = u_2(\omega_1 + \omega_2 - \tilde{x}_1^1)\}.$$

Obviously, that in Edgeworth box for 2nd agent indifference curve this point is placed strictly between points $\tilde{x}_1^1$, $\tilde{x}_1^2$. Further find neighborhoods V_1, V_2 of points $\tilde{x}_1^1$, $\tilde{x}_1^2$ and a neighborhood V_{max} of point x_1^{max} satisfying the following conditions:

1) At every point from V_{max} the 1st agent utility *is strictly more* than his/her utility at *any* point from neighborhoods V_1, V_2;

2) For every point from V_1 if the trajectory passes through this point (*i.e.* it is starting from this point as a point of initial data in Cauchy problem) then it certainly passes through some point of neighborhood V_{max}.

Clearly that such neighborhoods can be found, since first it is possible to find neighborhoods satisfying 1) and then if necessary to reduce neighborhoods V_1, V_2. However now we come to the contradiction because 1st agent utility is non-monotonically changed in the part of trajectory where 2nd agent is passive — it contradicts to the property that every contract defined by trading rule is mutually beneficial.[31]

8. LOCAL STABILITY IN CONTRACTUAL PROCESSES

In this section we consider the problem of local stability of contractual processes. But first of all we recall classical definitions.

Let's consider some autonomous differential equation $\dot{x} = F(x)$ and let $\bar{x}$ be its stationary (or critical, equilibrium) point, *i.e.* a point for which $F(\bar{x}) = 0$.[32]

A stationary point $\bar{x}$ is called locally stable if for each neighborhood of this point it is possible to find (another) neighborhood such that the solution of our equation starting from any point of last neighborhood (initial data in neighborhood) will never leave the limits of first neighborhood. Formally:

$$\forall \varepsilon > 0\ \exists \delta > 0: \text{ if } \|x(0) - \bar{x}\| < \delta, \text{ then } \|x(t, x(0)) - \bar{x}\| < \varepsilon \ \ \forall t \geq 0,$$

[31] One can yield a contradiction by a faster way: it is enough to notice, that on a ray, outgoing from ω_1 and passing through the point $\tilde{x}_1^2$, the point of 1st agent utility maximum should settle down "closer" to the point ω_1, rather than similar maximum point for 2nd utility. It follows from the fact that points $\tilde{x}_1^1$, $\tilde{x}_1^2$ are located on a common 1st agent indifference curve, *i.e.*, $u_1(\tilde{x}_1^1) = u_1(\tilde{x}_1^2)$. But it means that $\tilde{x}_1^1 \neq \tilde{x}_1^2$ is impossible.

Another way to obtain a contradiction is to notice, that a trajectory, starting from a neighborhood of point $\tilde{x}_1^2$ never can get in a neighborhood of $\tilde{x}_1^1$, since for this it has "to overcome" an area of point x_1^{max} neighborhood, in which the vector field of trading rule is oppositely directed.

[32] For differential inclusion $\dot{x} \in F(x)$ stationarity condition has the form $0 \in F(\bar{x})$. However by virtue of specific properties of inclusion $\dot{x} \in F(x)$ corresponding to contractual processes condition $0 \in F(\bar{x})$ is equivalent to $\{0\} = F(\bar{x})$.

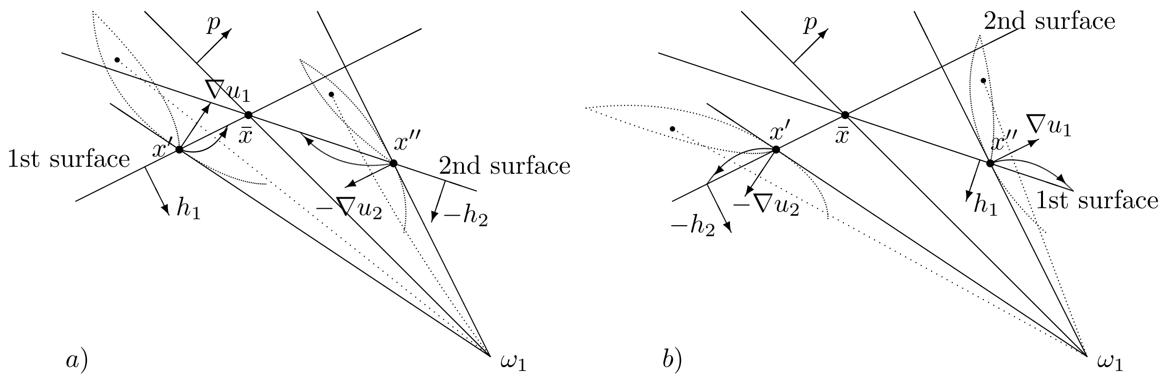

Fig. 7. Locally stable $a)$ and unstable $b)$ equilibrium $\bar{x}$

where $x(t, x(0))$ is the solution of the Cauchy problem with an initial point $x(0) \in \mathrm{dom}F$. However for our case it is not sufficient to have only ordinary stability it is also necessary that in the limit current allocation becomes equilibrium one (because we are interested in converging processes). Formally, in addition it is necessary to require the following:

$$\lim_{t\to+\infty} x(t, x(0)) = \bar{x}, \quad \forall\, x(0): \quad \|x(0) - \bar{x}\| < \delta.$$

Thus, speaking about local stability of contractual process we mean local asymptotical stability. Similar definitions are given for the case of differential inclusions.

As usual, we begin our analysis from the study of 2×2 economy case. It is rather simple case and one can obtain a characterization of locally stable equilibria already from the analysis of diagrams 7 $a)$, $b)$. In these diagrams (in the style of Edgeworth box) instead of projection onto agents' maximal surfaces there is shown the projection onto tangent hyperplane to maximal surface at equilibrium point $\bar{x}$ that is correct for the local analysis. The only difference between cases $a)$ and $b)$ is actually that we *have renumbered* the tangent hyperplanes determined by vectors h_i, $i = 1, 2$ calculated at a point $\bar{x}$. However this changed the case radically: locally stable equilibrium has turned into unstable! As it is shown in Fig. 7 some current points x', x'' are approaching to equilibrium but after change of surfaces numeration they are moving off from equilibrium as it is shown in 7 $b)$. Thus one can notice that in local stability the key role is played by an inter-location of vectors h_1, h_2 and p concerning equilibrium point. Moreover this allows us to formulate a hypothesis for the work: an equilibrium is stable if (locally) every current consumption of *active* individual *is strictly less* preferable of his/her consumption in equilibrium. This hypothesis is fulfilled for the situations similar described in diagram 7 $a)$ and is violated for 7 $b)$. For 2×2 economy it is possible strictly to prove the validity of our hypothesis. However already for the 3 goods economy this hypothesis becomes rather doubtful.

Really, the considerations below show that for economy with two agents and 3 commodities our hypothesis can be true only in some degenerated cases.

Let's consider economy with two individuals and with commodity space of dimension $l \geq 3$. Let $x \in \mathbb{R}^l_+$ denotes the 1st individual consumption and $y \in \mathbb{R}^l_+$ is applied to denote 2nd agent consumption and let preferences be defined via Cobb–Douglas functions:

$$u_1(x) = \prod_{j=1}^{l} x_j^{\alpha_j}, \quad \alpha \gg 0, \; \sum \alpha_j = 1, \quad u_2(y) = \prod_{j=1}^{l} y_j^{\beta_j}, \quad \beta \gg 0, \; \sum \beta_j = 1.$$

Let $\omega^i \gg 0$, $i = 1, 2$ be some initial endowments. It is well known, that in such economy there is an unique equilibrium which we denote by $(\bar{x}, \bar{y})$. Let's show that in *every* neighborhood of this equilibrium there are points (x, y) (allocations) such that an individual, say 1st, is active and the 2nd individual is passive, *i.e.* $\langle \nabla u_1(x), x - \omega^1 \rangle = 0$ & $\langle \nabla u_2(y), y - \omega^2 \rangle > 0$, and in addition $u_1(x) > u_1(\bar{x})$.

With this purpose let us any allocation $(\tilde{x}, \tilde{y})$ such that only 1st individual is active and $u_1(\tilde{x}) > u_1(\bar{x})$. Such allocations in 3 goods economy do exist (computer finds them in particular examples) though they are absent in two-goods economy. Further, reasoning in Edgeworth we can connect by linear segment points $\tilde{x}$ and $\bar{x}$ and show, that the points from the segment interior are strictly inside of maximal surface. It follows from the fact, that for Cobb–Douglas functions the restriction $\langle \nabla u_1(x), x - \omega^1 \rangle \geq 0$ is a restriction defined by concave function (thus the set of allocations limited by the maximal surface is convex). Really, for simplicity taking the logarithm of utility and calculating a gradient, we find

$$\langle \nabla \ln u_1(x), x - \omega^1 \rangle = 1 - \sum \frac{\alpha_j \omega_j^1}{x_j},$$

where in the right hand part of equality a strict concave function is written down. It follows from this that if from the point of initial endowments ω^1 to let out a ray going through some point of segment $x(\gamma) = \gamma\tilde{x} + (1 - \gamma)\bar{x}$, $\gamma \in (0, 1)$ then the point of intersection of the ray and the 1st agent maximal surface is placed on the ray further than point $x(\gamma)$. Thus, the 1st agent utility at a point of intersection is greater than his/her utility at a point $x(\gamma)$, which in turn is more than his/her utility at a point of equilibrium. So, we have shown that in all points which are located on a continuous curve on the maximal surface, it is constructed as all points of its intersection with all designed rays, the 1st agent utility is strictly more than utility at a point of equilibrium. It denies our working hypothesis since if it is incorrect already for Cobb–Douglas utilities then in general it cannot be considered acceptable.

So, in formulated above form our characteristic property for an equilibrium to be locally stable relative to general contractual processes is unsatisfactory. One of opportunities to elaborate positive result is to relax the stability requirement and to require (local) convergence not for any rule of trade but for the rules having some additional and economically reasonable properties. With this in mind we first try to understand better what is the basic distinction between 2-goods and 3-goods economies.

In 2-goods case and for a small enough neighborhood of an equilibrium point, if the current allocation is on the maximal surface, then any mutually beneficial exchange between the agents with necessity (almost always) involves the break of gross contract, *i.e.*, contractual process goes with obligatory break of contracts. Differently, if the current point of a trajectory is placed in a neighborhood of stable equilibrium then process further goes according to alternative (**i**) of previous paragraph: when time elapsed there can be only one active individual and his/her current consumption is strictly less preferable of equilibrium one (hence, using results of the previous paragraph one can easy establish convergence of a trajectory to equilibrium). This statement is true as soon as for a small enough neighborhood of equilibrium the normalized gradients of utility functions are almost equal to the vector of equilibrium prices but the vectors h_i, $i = 1, 2$ being calculated at equilibrium are disproportionate to equilibrium prices. However already in 3-goods economy it is not so, and in any neighborhood of equilibrium one can find an allocation on the maximal surface admitting a mutually beneficial exchange without break of contracts, *i.e.*, it is possible Pareto–improvement without break! This motivates the following

Definition 8.1. *A rule of trade $v : \mathcal{A}(X) \to L^c$ is called* ***benevolent****, if $v(x)$ does not attract the break of contracts in all situations when a mutually beneficial exchange without break is possible.*

Contractual UB–process is called ***benevolent****, if it is defined by a benevolent rule of trade.*

In substantial terms Definition 8.1 means that before the individuals sign a new contract, they carefully investigate opportunities for a mutually beneficial exchange being aimed to find a contract without subsequent break of made earlier contracts. The signing of a contract with subsequent break is carried out only if there is no any other opportunity to get an agreement.

Formally, for process by Definition 4.1 the concept of a benevolent rule of trade requires performance of the following conditions.

Let $x \in \mathcal{A}(X)$ be some allocation stable relative to the break of aggregated contract $x - \omega$ and let

$$\mathcal{I}^a(x) = \{i \in \mathcal{I} \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0\} \neq \emptyset$$

be *nonempty* set of all active individuals. Let us define

$$W^{fr}(x) = \{w \in L^c \mid \langle \nabla u_i(x_i), w_i \rangle > 0, \ \forall\, i \in \mathcal{I} \ \& \ \langle h_i, w_i \rangle > 0, \ \forall\, i \in \mathcal{I}^a(x)\}. \tag{8.1}$$

This is the set (possible empty) of all mutually beneficial contracts that being signed do not attract the break of aggregated contract $x - \omega$.[33] If $W^{fr}(x) \neq \emptyset$ then

$$\langle h_i, v_i(x) \rangle \geq 0, \quad \forall\, i \in \mathcal{I}^a(x) \ \ \& \ \ \langle \nabla u_i(x_i), v_i(x) \rangle > 0, \quad \forall\, i \in \mathcal{I}. \tag{8.2}$$

The following statement gives some criterion of local stability in economy with two agents.

Proposition 8.1. *Let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be an* ***isolated*** *equilibrium allocation in economy with two agents and let standard assumptions be satisfied. Then if for some neighborhood $V_{\bar{x}}$ in $\mathcal{A}(X)$ of point $\bar{x}$ for every $y \in V_{\bar{x}}$ satisfying $\langle \nabla u_i(y_i), y_i - \omega_i \rangle \geq 0$, $i = 1, 2$ the following relation*

$$[\exists\, j : \langle \nabla u_j(y_j), y_j - \omega_i \rangle = 0 \ \ \& \ \ W^{fr}(y) = \emptyset] \quad \Rightarrow \quad u_j(y_j) \leq u_j(\bar{x}_j) \tag{8.3}$$

holds, then equilibrium is locally stable relative to (locally) benevolent contractual processes.

Note that in the left hand part of relation (8.3) it is described a situation for which a mutually beneficial exchange without the subsequent break is impossible. Thus, the requirement (8.3) tells us that in such cases an active individual prefers equilibrium consumption to current one; the last can also be written down as $\langle \nabla u_j(y_j), \bar{x}_j - y_j \rangle > 0$ for $\bar{x}_j \neq y_j$ (it follows from strict concavity of utility). The proof of Proposition 8.1 is placed in Appendices.

Remark 8.1. Notice that for 2-goods economy an equilibrium $\bar{x}$ is isolated if vectors h_1, h_2 calculated at the equilibrium point are non-collinear. Moreover under assumption (**S**) and if equilibrium is an interior point then it is possible to prove that any contractual process in 2×2

[33] Here $L^c = \{(v_1, \ldots, v_n) \in E^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}$ is the space of contracts. Remember that condition $\langle h_i(x), w_i \rangle \geq 0$ means that individual $i \in \mathcal{I}^a$ is not interested in the break of contract $x - \omega$ when w is signed at a state x.

economy is locally benevolent. To see this it is enough to note that for any allocation y on the maximal surface and from a neighborhood of equilibrium a cone of improvements

$$z \in \mathbb{R}^2 \mid \langle \nabla u_1(y_1), z\rangle > 0 \ \ \& \ \ \langle \nabla u_2(y_2), z\rangle < 0$$

does not intersect a half-plane $\langle h_1(y_1), z\rangle \le 0$ when 1st agent is active, or does not intersect a half-plane $\langle h_2(y_2), z\rangle \ge 0$ when 2nd agent is active. It will be so since in a neighborhood of equilibrium the utility gradients are close to the equilibrium prices and in view of $h_i(y_i) \approx h_i(\bar{x}_i)$, $\langle h_i(\bar{x}_i), \omega_i - \bar{x}_i\rangle > 0$, $i = 1, 2$.

Thus Proposition 8.1 quite corresponds with our previous reasonings and serves their formal proof. Moreover this allows us to formulate a computable criterion of local stability for equilibrium in economy 2×2.

In presented form it is not easy to extend the result of Proposition 8.1 to the case of any number of agents because the fact that economy has only two agents is essential in the proof. Really, if there are only two individuals and one of them is active in a current state from a suitable neighborhood of equilibrium then one can conclude that for benevolent trajectories the utility of passive agent is strictly more than his/her utility in equilibrium. If there are more than one passive individual then one can surely assert only that among them there is at least one individual with strictly greater utility than in equilibrium.

Despite of difficulties to study the general case of local stability, in the following section we shall continue investigation of benevolent proper-contractual processes. Under certain assumptions it will be proved that these processes generically converge to equilibrium.

9. CONVERGENCE OF BENEVOLENT UB-PROCESSES

We begin the analysis from detailed research of a general case of benevolent trajectories. With this in mind we first study situations in which a mutually beneficial exchange without break of contracts is impossible. The following lemma describes necessary conditions that such situation takes place for some allocation.

Lemma 9.1. *Let $x \in \mathrm{ri}\mathcal{A}(X)$ be an allocation stable relative to the partial break of gross contract $x - \omega$. Let standard assumptions be satisfied and let mutually beneficial exchange without the subsequent break of contracts is impossible,* i.e. $W^{fr}(x) = \emptyset$. *Then there exists a vector $p \neq 0$ so that for each individual $i \in \mathcal{I}$ there are numbers $\alpha_i \ge 0$, $\beta_i \ge 0$ such that*

$$p = \alpha_i \nabla u_i(x_i) + \beta_i h_i(x_i), \quad \forall\, i \in \mathcal{I} \tag{9.1}$$

and the following complementarity slackness conditions

$$\beta_i \cdot \langle \nabla u_i(x_i), x_i - \omega_i\rangle = 0, \quad \forall\, i \in \mathcal{I}$$

are fulfilled.

Complementarity slackness conditions appeared in this lemma just serve a convenient form to describe the fact that for each passive individual (if $\langle \nabla u_i(x_i), x_i - \omega_i\rangle > 0$) the value $\beta_i \ge 0$

should be zero while for an active agent (if $\langle \nabla u_i(x_i), x_i - \omega_i\rangle = 0$) it can be strictly more than zero. Further, it is easy to see that up to normalization of p the values α_i and β_i in the formula (9.1) are unequivocally defined since $\nabla u_i(x_i)$ and $h_i(x_i)$ are the non-collinear couple of vectors for any active individual in the current allocation. It allows us among the active individuals correctly to identify the agents who can really influence a course of contractual process via a break of the contracts.

So, we shall call an active individual i *really active*, if $\beta_i > 0$. Accordingly, if $\beta_i = 0$ for an active agent then he/she may be called as *fictitiously active* (locally in process behavior of these individuals similar to passive ones). Also let's name the individual *really passive*, if he/she is passive *or* fictitiously active.

Proof of Lemma 9.1. By definition (8.1) of the set $W^{fr}(x)$ one can equivalently to rewrite condition $W^{fr}(x) = \emptyset$ in the following way.

Let us define $B_i(x) = \{z \in E \mid \langle \nabla u_i(x_i), z\rangle > 0\}$ if the individual i is passive and let $B_i(x) = \{z \in E \mid \langle \nabla u_i(x_i), z\rangle > 0 \ \&\ \langle h_i(x_i), z\rangle > 0\}$ for the active individual. Then

$$\prod_{\mathcal{I}} B_i(x) \cap L^c = \emptyset,$$

where, remember $L^c = \{(v_1, \ldots, v_n) \in E^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}$ is the space of contracts. Notice that $B_i(x) \neq \emptyset$ for active i since $\nabla u_i(x_i) \neq 0$, $\langle \nabla u_i(x_i), x_i - \omega_i\rangle = 0$ and $\langle h_i, x_i - \omega_i\rangle < 0$ (hence vectors are non-collinear), and of course $B_i(x) \neq \emptyset$ for passive agents. Therefore one can apply separation theorem and find $\pi = (p_1, \ldots, p_n) \neq 0$ such that

$$\langle \pi, \prod_{\mathcal{I}} B_i(x)\rangle \geq \langle \pi, L^c\rangle.$$

We see that functional π is bounded from above on subspace L^c which is possible only if $\langle \pi, L^c\rangle = 0$ and, therefore, in standard manner one can conclude that $p_i = p_j$, $\forall\, i \neq j$. Denote $p = p_i \neq 0$. Further, it is easy to see that $\langle \pi, \prod_{\mathcal{I}} B_i(x)\rangle \geq 0$ is possible only if (applying $\pi = (p, \ldots, p)$)

$$\langle p, B_i(x)\rangle > 0, \quad \forall\, i \in \mathcal{I}$$

is true. Thus, for every active i the inequality $\langle p, z\rangle > 0$ is a corollary of two inequalities: $\langle \nabla u_i(x_i), z\rangle > 0$, $\langle h_i(x_i), z\rangle > 0$, and for passive agent only of first of them. Now applying Farkas lemma (or again separation theorem) we conclude the existence of $\alpha_i \geq 0$, $\beta_i \geq 0$ demanded in the statement of lemma.

The property of a trajectory to be benevolent is rather qualified requirement which is applied to a rule of trade. In particular, it is easy to see that the vector p which existence was proved in Lemma 9.1 being normalized as $\|p\| = 1$ is continuous function of the current point of a trajectory.

Lemma 9.2. *Let $x(t)$, $t \geq 0$ be a benevolent UB-trajectory by Definition 4.1 and let standard assumptions be satisfied. Then the vector $p = p(x(t))$, $\|p\| = 1$, existing by Lemma 9.1 at all points of trajectory $x(t)$ where beneficial exchange without contracts breaking is impossible is continuous function in its domain.*

It is clear, that in lemma conditions when there is the specified continuous dependence of a vector p from the current point of a trajectory, the same thing can be said about the coefficients α_i, β_i of decomposition (9.1): they are also continuously depend on time. It follows from the fact (already mentioned) that vectors ∇u_i, h_i being calculated at required points are non-collinear.

Proof of Lemma 9.2. It follows from the continuity of a trajectory $x(t)$ that for each individual i the set of all moments $t > 0$ where he/she is passive is open on interval $(0, +\infty)$, because the set is defined via condition $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle > 0$. Due to Lemma 9.1 we have $p(x(t)) = \nabla u_i(x_i(t))/\|\nabla u_i(x_i(t))\|$ in every moment t where individual i is passive. Therefore, since the gradient of utility continuously depends on trajectory points one can conclude that as the function of time $p(t)$ changes continuously in a neighborhood of t. Now we need to show that if $\dot{x}(t) \neq 0$, *i.e.*, if the point $x(t)$ is not equilibrium, and if the mutually beneficial exchange without break is impossible then a passive individual does exist. Assuming that all individuals are active via (9.1) (the mutually beneficial exchange without break is impossible) we have

$$p(t) = \alpha_i \nabla u_i(x_i(t)) + \beta_i h_i(x_i(t)), \quad \forall\, i \in \mathcal{I}.$$

Further let us multiply these equality on vectors $x_i(t) - \omega_i \neq 0$ and then sum the received equalities. As a result, since from the activity of the individuals we have $\langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0$, $\forall\, i \in \mathcal{I}$ and due to $\sum (x_i(t) - \omega_i) = 0$ we obtain

$$0 = \sum \beta_i \langle h_i(x_i(t)), x_i(t) - \omega_i \rangle.$$

Since $\beta_i \geq 0$ and $\langle h_i(x_i), x_i - \omega_i \rangle < 0$, $\forall\, i \in \mathcal{I}$, then the last equality is possible only if $\beta_i = 0$ for all i, that is possible only in equilibrium. It is a contradiction.

Further we will be interested in some specific properties of proper-contractual trajectories by Definition 4.1 (not necessarily benevolent!). In fact, we need to clear those situations, when at the points of a trajectory more than one active individual may exist. With this in mind we remind that at the current point of a trajectory $x(t)$ the measure of break of the gross contract is defined as a minimum (provided that it is less than zero, otherwise a break does not occurs) of some values determining desirable break for the active individuals. Desirable break of gross contract for the agent i is defined by value

$$\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \omega_i - x_i(t) \rangle},$$

see Lemma 4.1. However for two individuals simultaneously define the size of break of the contracts in *nearest subsequent* after t moments of time, it is necessary that the measure of desirable break of gross contract coincides with a general minimum and, therefore, both measures should coincide among themselves. This motivates the following definition.

Definition 9.1. *A contractual trajectory $x : [0, +\infty) \to \mathcal{A}(X)$ (process) is called **non-degenerate** if for all non-equilibrium points $x(t)$ of its hit on maximal surface for each couple of active individual i, j, $i \neq j$ (*i.e. *if $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i \rangle = \langle \nabla u_j(x_j(t)), x_j(t) - \omega_j \rangle = 0$) the following inequality*

$$\lambda_i(x(t), v(x(t))) = \frac{\langle h_i(x_i(t)), v_i(x_i(t)) \rangle}{\langle h_i(x_i(t)), \omega_i - x_i(t) \rangle} \neq \frac{\langle h_j(x_j(t)), v_j(x_j(t)) \rangle}{\langle h_j(x_j(t)), \omega_j - x_j(t) \rangle} = \lambda_j(x(t), v(x(t)))$$

holds.

A trading rule is called ***non-degenerate*** *relative to initial endowments allocation* $\omega = (\omega_1, \ldots, \omega_n)$ *if a generated trajectory with initial data* $x(0) = \omega$ *is non-degenerate.*

Non-degenerate contractual trajectories are easier in the analysis, because in each moment of time only one individual sets a measure of break of gross contract, *i.e.* he/she can be considered to be as an active "leader" of contractual process. However, for non-degenerate processes of a general form when time elapses there can be a change of the leader. Below we will see that it does not occur in case of benevolent trajectories.

With this in mind we first show, that for *any not stabilized* trajectory the set of those moments where there are two or more active agents has a structure similar to discrete one[34].

Lemma 9.3. *Let* $x(t)$ *be a non-degenerate UB-contractual trajectory by Definition 4.1 and 9.1. Let in the moment* τ *the derivative* $\dot{x}(\tau) \neq 0$ *and let an agent* i *is active,* i.e. $\langle \nabla u_i(x_i(\tau)), x_i(\tau) - \omega_i \rangle = 0$. *Then for some* $\varepsilon > 0$ *and for all points from* $(\tau, \tau + \varepsilon)$ *only one of the following alternatives takes place:*

(**i**) *If* $\lambda_i(x(\tau), v(x(\tau))) = \lambda^{min}(x(\tau), v(x(\tau))) < 0$ ***only*** *individual* i *is active and all other agents are passive.*

(**ii**) *If* $\lambda_i(x(\tau), v(x(\tau))) = \lambda^{min}(x(\tau), v(x(\tau))) = 0$ *individual* i ***may be*** *active but it is not certainly the case however* ***all other*** *agents are certainly* ***passive****.*

(**iii**) *If* $\lambda_j(x(\tau), v(x(\tau))) > \lambda^{min}(x(\tau), v(x(\tau))) = 0$ *for each active individual then in interval* $(\tau, \tau + \varepsilon)$ ***all agents are passive****.*

One can find the proof of Lemma 9.3 in Appendices. The statement of this lemma implies the following important

Corollary 9.1. *If a contractual trajectory is non-degenerated and* $\dot{x}(t) \neq 0$, $\forall\, t \geq 0$, *then the set of all time moments where the number of active individuals can be two or greater is no more than enumerable.*

Remark 9.1. The statement of Lemma 9.3 may seem almost obvious however it is necessary to remember that we deal with the solution of a differential equation with a discontinuous right hand part and, therefore, this solution is not obliged to be differentiable function. Therefore appropriate analysis should be realized with the special carefulness.

It is seemed, that the result of lemma can be extended to a general case of not necessarily non-degenerate trajectories. Certainly, it has to be another edition in which possible such variant of alternative (**i**): from the set all active individuals satisfying to this alternative condition there may be separated the group of the individuals each of them is active on some open interval of time directly contiguous to the considered time moment. However the strict proof of this fact is not presented while... As a consequence to generalized lemma one can hope to receive such fact: the number of the moments of time, when the set of the active individuals is reconstructed, is no more than enumerable.

[34] Strictly speaking, this set can have limit points, however it does not influence the subsequent analysis.

Further we show that while equilibrium is not attained along a benevolent trajectory there is only one individual that can be "really" active. Remember that is we have named really active such active individual which in the relation (9.1) from Lemma 9.1 has parameter $\beta_i \neq 0$.

Theorem 9.1. *Let $x(t)$ be non-degenerate benevolent trajectory and let standard assumptions be fulfilled. Then only one of the following alternatives can be true:*

(**i**) *There are no (almost) time moments on the interval $[0, +\infty)$ when the breaking of gross contract $x(t) - \omega$ is realized,* i.e., *for almost all time moments during contractual process all individuals are passive.*

(**ii**) *There exists such time moment $\tau > 0$ that for almost all time moments on the interval $[0, \tau)$ all individuals are passive and the contrary at the moment τ: all individuals are active,* i.e., *$x(\tau)$ is an equilibrium.*

(**iii**) *There exist time moments $\tau_1 > 0$ and $\tau_2 > \tau_1$ such that for almost all time moments on the interval $[0, \tau_1)$ all individuals are passive and there is the only real active individual on the interval $[\tau_1, \tau_2)$ and if $\tau_2 \neq +\infty$ then at the moment τ_2 all individuals are active,* i.e., *$x(\tau_2)$ is an equilibrium.*

Theorem 9.1 describes rather important properties of non-degenerate benevolent trajectories which allow us to conclude the convergence of this type trajectories to an equilibrium.

Corollary 9.2. *Let the standard assumptions be fulfilled. Then any benevolent rule of trade generating non-degenerate contractual process defines proper-contractual UB-trajectory, for which all limit points are equilibria. Thus non-degenerate benevolent processes are quasi-globally stable.*

Proof of Corollary 9.2. If alternative (**ii**) or (**iii**) when $\tau_2 < +\infty$ are realized then the convergence of a contractual trajectory to equilibrium is obvious. If the alternative (**i**) or (**iii**) with $\tau_2 = +\infty$ are realized then we are in the condition of alternative (**i**) from § 7.1 (see page 44) and now we can apply Lemma 7.1 and Remark 7.1 that proves equilibrium properties of any limit point.

Proof of Theorem 9.1. Let us determine a time moment $\tau > 0$ as a first moment when the mutually beneficial exchange without partial break of gross contract $x(t) - \omega$ is impossible. If there are no such moments then alternative (**i**) is realized. However, if the set of all such moments is not empty then being the closed set it always has the minimal element. Therefore moment τ is determined correctly. Further, if $\dot{x}(\tau) = 0$ then $x(\tau)$ is an equilibrium and, therefore, the alternative (**ii**) is realized. Let $\dot{x}(\tau) \neq 0$. Now we are able to apply Lemma 9.1 and can conclude that at the moment τ there is at least one really active individual. Really, otherwise $\beta_i(\tau) = 0$ for all $i \in \mathcal{I}$ that is possible only in equilibrium. Let $\mathcal{I}^{ra}(\tau) \subset \mathcal{I}$ be nonempty set of all really active individuals at the moment τ. Further one can apply the fact that our trajectory is non-degenerate and show that alternative (**i**) of Lemma 9.3 is true. We need to prove that $\lambda_i(x(\tau), v(x(\tau))) < 0$ for some active individual $i \in \mathcal{I}$ at the moment τ. To do it we need to show $\langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle < 0$. With this in mind one can apply Lemma 9.1 and conclude the existence of a vector $p(\tau) \neq 0$ and, for each i, numbers $\alpha_i(\tau) \geq 0$, $\beta_i(\tau) \geq 0$ such that (9.1) is carried out:

$$p(\tau) = \alpha_i(\tau)\nabla u_i(x_i(\tau)) + \beta_i(\tau)h_i(x_i(\tau)), \quad \forall\, i \in \mathcal{I}.$$

Further, for each really active individual i from $\mathcal{I}^{ra}(\tau) \neq \emptyset$ multiply the appropriate equality on vector $v_i(x_i(\tau))$ and then sum the received equalities. The obtained result can be written down as

$$\langle p(\tau), \sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau))\rangle - \sum_{\mathcal{I}^{ra}(\tau)} \alpha_i \langle \nabla u_i(x_i(\tau)), v_i(x_i(\tau))\rangle =$$
$$= \sum_{\mathcal{I}^{ra}(\tau)} \beta_i(\tau)\langle h_i(x_i(\tau)), v_i(x_i(\tau))\rangle.$$

Since each summand in the right hand part of this equality has a positive factor (strictly more than zero) we shall receive required property if it will be established that the value in the left hand part of equality is negative. But it is so because by the definition of contract $\sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau)) = -\sum_{\mathcal{I}\setminus\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau))$ that in view of its mutual benefit for really passive individuals gives

$$\langle p(\tau), \sum_{\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau))\rangle = -\langle p(\tau), \sum_{\mathcal{I}\setminus\mathcal{I}^{ra}(\tau)} v_i(x_i(\tau))\rangle < 0.$$

Therefore, the left hand part of previous equality is the summation of negative and non-positive values and as a whole it is negative.

So, at present moment we have proven that alternative (**i**) of Lemma 9.3 is realized. This implies that for some $\varepsilon > 0$ on the interval $(\tau, \tau+\varepsilon)$ the contractual process goes with a break of gross contract and only one agent is active. Let i_0 be this individual. Only this individual (from complementarity slackness conditions from Lemma 9.1) can be *really* active on the interval $(\tau, \tau+\varepsilon)$ and, therefore,

$$\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall\, j \neq i, \quad \forall\, t \in (\tau, \tau+\varepsilon).$$

At last, applying Lemma 9.2 we can in these relations pass to limit for $t \to \tau + 0$ (all functions are continuous) concluding that $\beta_{i_0}(\tau) > 0$ and $\beta_j(\tau) = 0$, $\forall\, j \neq i_0$. Thus, the individual i_0 is *sole* really active agent on the interval $[\tau, \tau+\varepsilon)$.

Below, on former assuming that $x(\tau)$ is not equilibrium we prove the validity of alternative (**iii**).

With this in mind we first show, that for every $t > \tau$ the mutually beneficial exchange without partial break of gross contract $x(t) - \omega$ is impossible. Assuming opposite, find $t' > \tau$ as *infimum* of all moments where the exchange without break is possible. It is clear, that the set of all such moments forms an open set on $(\tau, +\infty)$ and t' can not belong to it. Therefore, at the moment t' the mutually beneficial exchange without break is impossible. Besides $x(t')$ can not be an equilibrium since the exchange goes after moment t'. Now we can apply Lemmas 9.1, 9.3 and reasoning similar to described above we can conclude the existence of $\delta > 0$ such that at *every* point of interval $[t', t'+\delta)$ the contractual process is realized with partial break of contracts. In a result we come to the contradiction with a choice of the moment t'.

Further we define $\tau_1 = \tau$ and find the moment τ_2 as infimum of all those moments of time from $(\tau_1, +\infty)$ when there is at least one another active individual distinct from i_0. If there are no such moments then $\tau_2 = +\infty$ and everything is proven. Let us assume $\tau_2 < +\infty$ and show that

$x(\tau_2)$ is an equilibrium. First of all note that at the moment τ_2 only individual i_0 can be really active. Really at this moment the mutually beneficial exchange without break is impossible and on the interval (τ_1, τ_2) only i_0 is active, therefore, applying Lemma 9.1 we obtain

$$\beta_{i_0}(t) > 0, \quad \beta_j(t) = 0, \quad \forall\, j \neq i_0, \quad \forall\, t \in (\tau_1, \tau_2).$$

Further, in view of Lemma 9.2 all functions $\beta_i(\cdot)$, $i \in \mathcal{I}$ are continuous on $[\tau_1, \tau_2]$ and, passing to limits by $t \to \tau_2 - 0$ we conclude

$$\beta_{i_0}(\tau_2) \geq 0, \quad \beta_j(\tau_2) = 0, \quad \forall\, j \neq i_0.$$

Further, if $\beta_{i_0}(\tau_2) \neq 0$ we are in conditions of alternative (**i**) from Lemma 9.3 and hence for some $\varepsilon > 0$ on interval $(\tau_2, \tau_2 + \varepsilon)$ does exist only one active individual. By the choice τ_2 this individual can not be i_0. Therefore i_0 is passive on $(\tau_2, \tau_2 + \varepsilon)$ and once again via Lemma 9.1 we conclude $\beta_{i_0}(t) = 0$ on the interval $(\tau_2, \tau_2 + \varepsilon)$. However $\beta_{i_0}(t)$ is continuous function on $(\tau_2, \tau_2 + \varepsilon)$ by Lemma 9.2. Now passing to a limit by $t \to \tau_2 + 0$ we conclude $\beta_{i_0}(\tau_2) = 0$. The received contradiction proves that $\beta_{i_0}(\tau_2) = 0$. However above it was established that $\beta_j(\tau_2) = 0$, $\forall\, j \neq i_0$. This is possible only for equilibrium point (since at the point $x(\tau_2)$ gradients of all individuals are pairwise collinear this is Pareto optimum which is also stable relative to the partial break of gross contract). Theorem 9.1 has proven.

One of our main results is presented in the following theorem on convergence of non-degenerate benevolent UB-processes.

Theorem 9.2. *Let $\mathcal{E}$ be a regular economy and the standard assumptions are fulfilled. Then any non-degenerate benevolent UB-processes converges to an equilibrium.*

As a corollary of this theorem applying Thom's theorems on density and openness of transversal sections it seems possible to prove the following result on generic convergence of benevolent contractual processes to an equilibrium.

Corollary 9.3. *For almost all economies of C^2-class every benevolent contractual UB-process converges to an equilibrium.*

Proof of Theorem 9.2. Theorem 9.1 and its Corollary 9.2 can be applied in the conditions of this theorem. Thus, each limit point of a trajectory is an equilibrium. Further we show, that in conditions of Theorem 9.2 every benevolent trajectory can have only one limit point.

Assume contrary and let x, y be two *different* limit points of a trajectory. Let's consider a linear segment with the ends x, y, *i.e.*, the set $\{\gamma x + (1-\gamma)y \mid 0 \leq \gamma \leq 1\}$. Across each point $z(\gamma) = \gamma x + (1-\gamma)y$, $\gamma \in (0,1)$ of the segment one can conduct a hyperplane so that points x, y are *strictly* in the different half-spaces. For example, such hyperplane $H(\gamma)$ can be conducted as a hyperplane which has $y - x$ as a vector of its normal. It is clear, that in such manner we can define a family of pairwise-not-crossed hyperplanes depending on parameter $\gamma \in (0,1)$ such that our *two different* limit points of trajectory are placed *in two different* open half-spaces defined by $H(\gamma)$. Hence, when time elapses the trajectory crosses every hyperplane infinite number of times and any limit point of these points of crossing is also a limit point of trajectory and, therefore, this is an equilibrium. Thus, the economy has a continuum of different equilibria, since for different γ we have limit points from different parallel hyperplanes. However each regular economy has a finite number of equilibria. This contradiction proves that there is the only limit point and, hence, benevolent UB-contractual process converges to an equilibrium.

10. EXAMPLES OF CONTRACTUAL PROCESSES: CONVERGENCE AND CYCLING

In this section[35] we consider some additional examples revealing the character of contractual processes. Besides there are described computer programs and results of computer modelling.

10.1. Convergence of UB-processes in an 2×3 economy

In this item we describe a computer program simulating proper-contractual process for 2×3 economy and also one numerical example of work of this program is presented.

The model of economy with 2 agents and 3 goods is considered, in which the preferences are defined by Cobb–Douglas functions in the logarithmic form:

$$u_1(x) = a_{11} \ln(x_1) + a_{12} \ln(x_2) + a_{13} \ln(x_3), \quad x \gg 0,$$

$$u_2(y) = a_{21} \ln(y_1) + a_{22} \ln(y_2) + a_{23} \ln(y_3), \quad y \gg 0.$$

On the start of the program there are determined the parameters of utility functions and initial endowments:

$$a = ((a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23})), \quad \omega = ((\omega_1^1, \omega_1^2, \omega_1^3), (\omega_2^1, \omega_2^2, \omega_2^3)).$$

There are also determined specific parameters: *step* > 0 (step) and *tochn* > 0 (closeness). Applying these data program finds an equilibrium and constructs a sequence of allocations $(x, y)^{(0)}, (x, y)^{(1)} \ldots, (x, y)^{(n)}$ corresponding to the rules of proper-contractual process (this one can see from algorithm), and represents the results graphically.

The only equilibrium is found from the following system of equations:

$$\omega_1^1 + \omega_2^1 = \bar{x}_1 + \bar{y}_1,\ \omega_1^2 + \omega_2^2 = \bar{x}_2 + \bar{y}_2,\ \omega_1^3 + \omega_2^3 = \bar{x}_3 + \bar{y}_1^3,$$

$$\nabla u_1(\bar{x}) \cdot \bar{x} = \nabla u_1(\bar{x}) \cdot \omega_1, \quad \nabla u_1(\bar{x}) = \alpha \cdot \nabla u_2(\bar{y}), \quad \alpha > 0.$$

Schema of program algorithm:

0. Program finds equilibrium.

1. $(x, y)^{(0)} := \omega$.

2. Program finds a mutually beneficial contract $v^{(n)} = (v_1, v_2)$ as follows:

   (*) in cube $[-0.5 \times \text{step}, 0.5 \times \text{step}]^3$ a point v_1 is randomly chosen relative to uniform distribution and $v_2 := -v_1$ is defined. If contract $v = (v_1, v_2)$ is mutually beneficial (*i.e.* $u_1(x^{(n)} + v_1) > u_1(x^{(n)})$ and $u_2(y^{(n)} + v_2) > u_2(y^{(n)})$) then **item 3**; otherwise return to (*).

3. $(x, y)^{(n+1)} := (x, y)^{(n)} + v^{(n)}$.

   If after signing of this contract the trajectory does not leave the limits the maximal surface then **item 4**. Otherwise this point is projected onto surface: from a linear segment $[(x, y)^{(n)} + v, \omega]$ a point inside area limited by maximal surface is chosen at a distance no more $0.01 \times$ step from the maximal surface. Then:

   $(x, y)^{(n+1)} :=$ the projected point.

[35] These examples and programs were constructed in collaboration with Sergey Kolbin, IV year student of MMF NSU (during 2005–2006 years I supervised his bachelor diploma).

Table 1. Discrete proper contractual trajectory: computation results of each 30th step for 1st agent

1st good $x_1^{(k)}$	2nd good $x_2^{(k)}$	3rd good $x_3^{(k)}$	break degree: $\frac{\|(x,y)^{(k+1)}-(x,y)^{(k)}\|}{\|v^{(k)}\|}$
9	9	2	ω_1
9.002	8.956	2.048	inside
8.234	8.802	2.647	inside
7.455	8.353	3.469	inside
6.785	8.092	4.285	inside
6.044	7.681	5.148	inside
5.167	7.187	6.021	inside
4.638	6.673	6.825	inside
3.743	6.347	7.705	inside
3.261	5.454	8.669	inside
2.642	4.876	9.398	0.845
2.552	4.420	9.410	0.532
2.396	4.218	9.416	0.726
2.345	3.756	9.426	0.668
2.384	3.384	9.433	0.831
2.370	3.119	9.437	0.334
2.259	2.955	9.441	0.470
2.184	2.813	9.444	0.404
2.119	2.676	9.446	0.841
2.027	2.611	9.447	0.953
1.918	2.610	9.448	0.949
1.880	2.545	9.449	0.945
1.925	2.397	9.451	0.824
1.882	2.368	9.451	0.943
1.844	2.360	9.452	0.129
1.761	2.258	9.454	equilibrium

4. Point $(x,y)^{(n+1)}$ becomes visible on monitor and point's parameters are written into the file.
5. If the distance from $(x,y)^{(n+1)}$ to equilibrium is more than step $\times$ tochn then **item 2**. Otherwise we think that trajectory arrives at equilibrium and program stops.

Table 1 reduces the results of the program work for the following numerical example. The preferences are defined by utility functions

$$u_1(x) = \ln(x_1) + \ln(x_2) + 9\ln(x_3), \quad u_2(y) = 9\ln(y_1) + 10\ln(y_2) + \ln(y_3).$$

Initial endowments: $\omega = (\omega_1, \omega_2) = ((9,9,2),(1,5,8))$. Then (the only) equilibrium allocation is $(\bar{x},\bar{y}) = ((1.761, 2.258, 9.454),(8.239, 11.742, 0.546))$.

In the program the parameters *step*=0.1, *tochn*=1 were given and one can see that a generated by the computer sequence of allocations $(x,y)^{(n)}$ "arrives" to equilibrium. Table 1 presents every 30th point of sequence $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$, *i.e.*, only the consumption of 1st agent is described: the amount of the first good consumed by 1st agent at each 30th step forms the first column, second column contains analogous results for 2nd good and so on. The estimations of the contracts break are specified in fourth column. These estimations are defined as a quotient of length of real progress of a trajectory to the length of signed contract, *i.e.*, written $\frac{\|(x,y)^{(n+1)}-(x,y)^{(n)}\|}{\|v^{(n)}\|}$ when the trajectory is moving by the maximal surface and it is applied "inside" if the current point is in the interior of area limited by maximal surface.

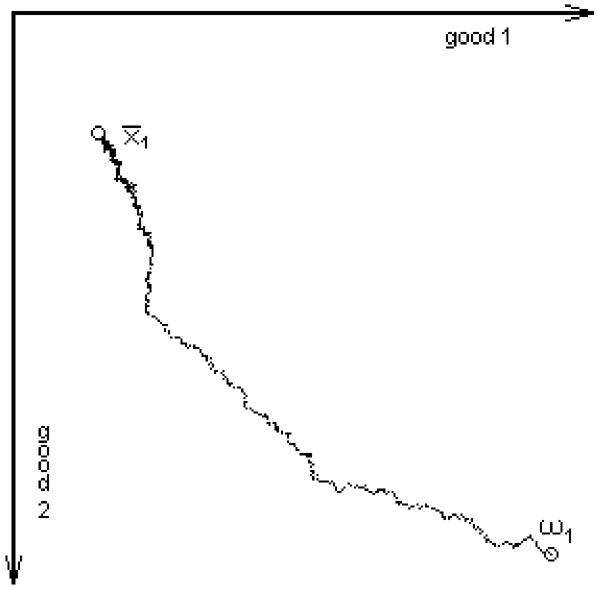

Fig. 8. Exchange dynamics between 1st and 2nd goods

Graphically the trajectory is presented as follows. Fig. 8 represents $(x_1(t), x_2(t))$ for each step of the program work. The grey colour is applied for the points inside the limits of maximal surface, black — the moving by the maximal surface. Similarly, the dynamics of pairs $(x_2(t), x_3(t))$ and $(x_3(t), x_1(t))$ is represented in Figures 9 and 10, accordingly.

10.2. Absence of convergence for 4×2 economy under assumptions CUB, IBA

Let us consider economy with 4 individuals which utilities define a family of indifference curves as it is presented in Fig. 12. At the starting time moment ($t = 0$) the current allocation coincides with the initial endowments: $x_1(0) = \omega_1 = (9, 1)$, $x_2(0) = \omega_2 = (3, 3)$, $x_3(0) = \omega_3 = (3, 9)$, $x_4(0) = \omega_1 = (3, 4)$. Further, let in a time interval $(0, t_1)$ the coalition $\{1, 2\}$ be active, in an interval (t_1, t_2) the coalition $\{2, 3\}$ is active, $\{3, 4\}$ is active in limits (t_2, t_3) and in an interval (t_3, t_4) the coalition $\{4, 1\}$ is active. Let's assume also that in the nearest future the order of coalition activity is: $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$ and that each time interval of activity is short enough. Suppose that for this interval of time a coalition have enough time only to break contracts signed in the past and have no time to sign a new contract. The specified modes of coalitions activity in time is visualized in Fig. 11.

Let further the signing of intra-coalition contracts is realized in the following way. On the segment $(0, t_1)$ a contract $v^{\{1,2\}} = (\overrightarrow{\omega_1 A_1}, \overrightarrow{\omega_2 A_2})$ is signed and realized. The bundles of goods for the first and second individuals at the moment t_1 are A_1 and A_2, accordingly. Further, on (t_1, t_2) a contract $v^{\{2,3\}} = (\overrightarrow{A_2 B_2}, \overrightarrow{\omega_3 A_3})$ is signed and realized by $\{2, 3\}$. Thus, the bundles of goods for 2nd and 3rd individuals at the moment t_2 are B_2 and A_3, accordingly. Further, on an interval (t_2, t_3) a contract $v^{\{3,4\}} = (\overrightarrow{A_3 B_3}, \overrightarrow{\omega_4 A_4})$ is realized by $\{3, 4\}$. So, the bundles of 3rd and 4th at the moment of time t_3 are B_3 and A_4, accordingly. At last, on an interval (t_3, t_4) a contract $v^{\{4,1\}} = (\overrightarrow{A_4 B_4}, \overrightarrow{A_1 B_1})$ is realized. In doing so the bundles of 4th and 1st at the moment t_4 become equal to B_4 and B_1, accordingly. Thus, at the moment of time t_4 the commodity bundle of agent i corresponds to the point B_i in the figure and the point A_i is the result of the

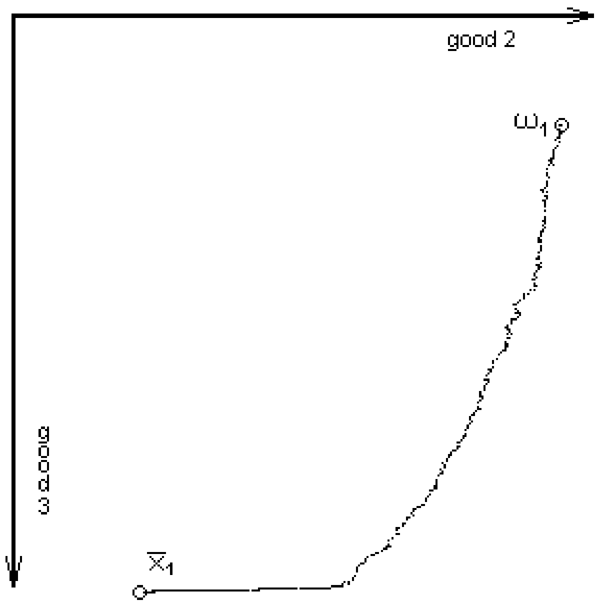

Fig. 9. Exchange dynamics between 2nd and 3rd goods

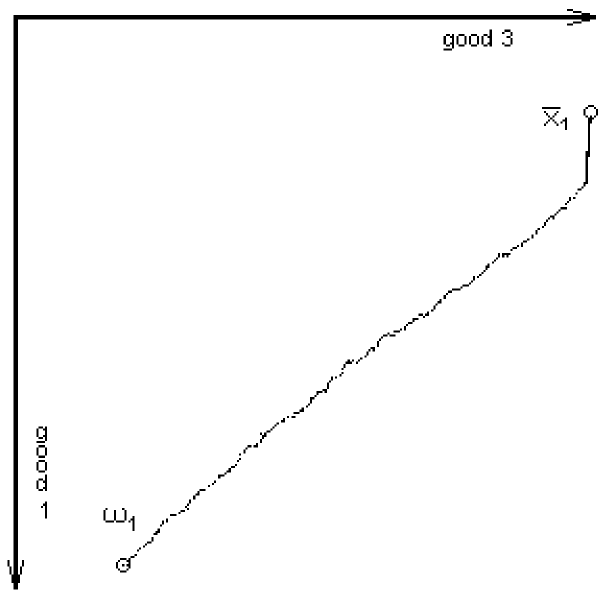

Fig. 10. Exchange dynamics between 1st and 3rd goods

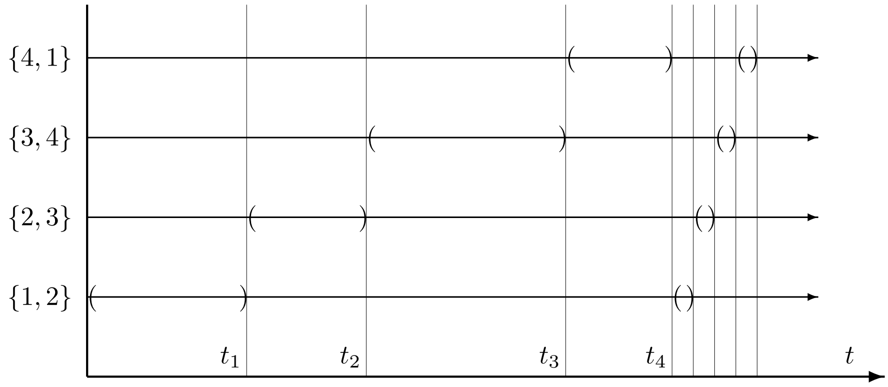

Fig. 11. Activity order for bilateral coalitions in 4×2 economy

first stage of barter exchange of the individual i: for each individual it is realized on a time interval and with an appropriate partner. In each presented diagram there are specified the coalitions which signed contract resulting the specified consumer program. One can also see from the figures that all above-stated contracts are mutually beneficial. Moreover, completing these diagrams if necessary it is possible to design appropriate indifference curves in such manner that each signing of new contract is carried out in accordance with principles of proper-contractual process, where in each stage of barter exchange the role of "new" endowments plays the sum of initial endowments with a flow of goods received from contracts signed him in *others* coalitions.

When all acts of described bilateral contracts are realized, at the moment t_4 the coalition $\{1,2\}$ is active again. However now 1st agent finds that it is favourable for him to partially break contract $v^{\{1,2\}}$ with 2nd agent and he/she completely breaks it raising his/her utility from consumption. As a result new commodity bundles of 1st and 2nd agents are the vectors corresponding to the points C_1 and C_2 in the figure, accordingly. In so doing since the interval of coalition activity is short, the agents have no time to sign a new contract. Next active coalition is $\{2,3\}$. Now 2nd agent completely breaks the contract $v^{\{2,3\}}$ signed him in the past in coalition $\{2,3\}$. Consumption bundles of 2nd and 3rd become ω_2 and C_3, accordingly. Then coalition $\{3,4\}$ is next to be active. Now 3rd agent observes that it is favourable for him to break off the contract $v^{\{3,4\}}$. As a result ω_3 and C_4 become the consumption bundles of 3rd and 4th, accordingly. Finally coalition $\{4,1\}$ becomes an active one. However once again there is an agent, now it is 4th, which desires to break off coalition contract $v^{\{4,1\}}$. The result of this is that ω_4 and ω_1 become 4th and 1st consumption bundles, accordingly.

Thus, one can see that the contractual trajectory has returned to the allocation of initial endowments and considered contractual process is cycled: the signing of last contract by coalition $\{4,1\}$ caused a breaking chain (likes avalanche) of all contracts! What can be said about this occasion? If the 1st individual were able to expect such development of events at a stage when he/she was signing contract with 4th agent or if at once after signing this contract he/she would limit appetites and has refused to break contract with 2nd agent, then the destruction of contractual structure of economy did not occur... However the behavior of such type should be clearly incorporated into the model of contractual process and this would mean essential modernization of our theoretical conceptions about proper-contractual processes.

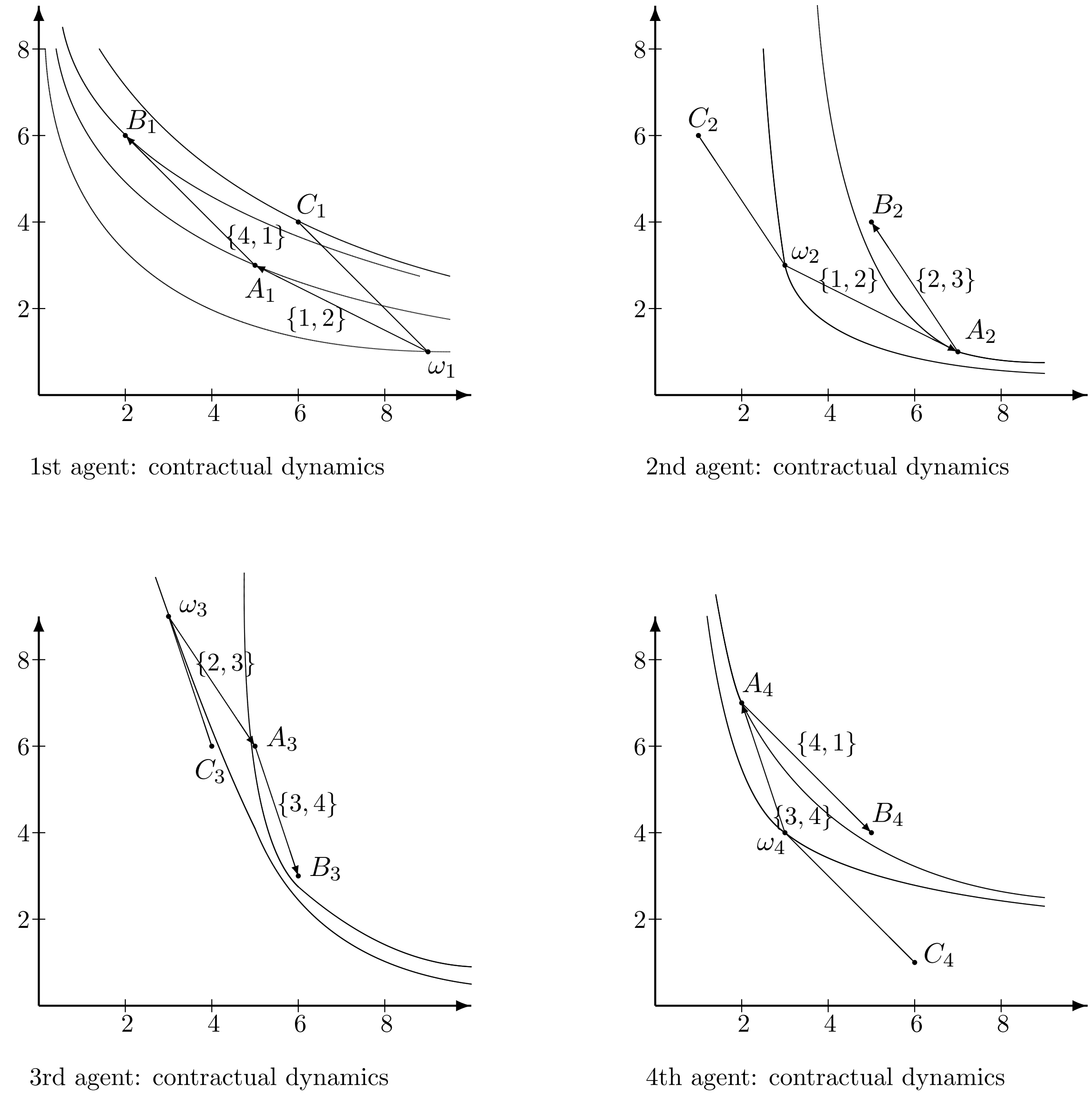

Fig. 12. An example of 4×2 economy where CUB-process is cycling and does not converge to equilibrium

10.3. Absence of convergence for 3×2 economy and UB-process with a piecewise-continuous trade rule

This example demonstrates the importance of assumption on continuity of a trade rule applied in the above analysis of proper-contractual UB-processes.

Let us consider an economy with 3 agents and 2 goods. In model there are defined an initial resources allocation $\omega = (\omega_1, \omega_2, \omega_3)$ and Cobb–Douglas utility functions in logarithmic form:

$$u_1(x) = a_{11}\ln(x_1) + a_{12}\ln(x_2), \quad x \gg 0,$$

$$u_2(y) = a_{21}\ln(y_1) + a_{22}\ln(y_2), \quad y \gg 0,$$
$$u_3(z) = a_{31}\ln(z_1) + a_{32}\ln(z_2), \quad z \gg 0.$$

Being defined model parameters one can find a sole equilibrium $\bar{\zeta} = (\bar{x}, \bar{y}, \bar{z})$ from the following system of equations:

$$\omega_1^1 + \omega_2^1 + \omega_3^1 = \bar{x}_1 + \bar{y}_1 + \bar{z}_1, \quad \omega_1^2 + \omega_2^2 + \omega_3^2 = \bar{x}_2 + \bar{y}_2 + \bar{z}_2,$$

$$\nabla u_1(\bar{x}) = \alpha \cdot \nabla u_2(\bar{y}) = \beta \cdot \nabla u_3(\bar{z}), \quad \alpha > 0,\ \beta > 0,$$
$$\nabla u_1(\bar{x}) \cdot \bar{x} = \nabla u_1(\bar{x}) \cdot \omega_1, \quad \nabla u_2(\bar{y}) \cdot \bar{y} = \nabla u_2(\bar{y}) \cdot \omega_2.$$

Further we will describe the program simulating proper-contractual process for 3×2 economy for the specified trade rule $w : \mathcal{A}(X) \to L^c = \{(v_1, v_2, v_3) \in \mathbb{R}^6 \mid v_1 + v_2 + v_3 = 0\}$ and then a numerical example of this program realization will be also considered.

On the start of the program there are determined the parameters of utility functions, initial endowments and an unambiguously defined trade rule. There is also determined a specific parameter *step* > 0. Applying these data program finds an equilibrium and constructs a sequence of allocations $\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(n)}$ (here $\zeta = (x, y, z)$) corresponding to the rules of proper-contractual process (this follows from algorithm), and represents the results graphically.

Schema of program algorithm:

0. Program finds equilibrium.

1. $\zeta^{(0)} := \omega$.

2. Program finds a mutually beneficial contract $w(\zeta^{(n)}) = (w_1, w_2, w_3)$ in accordance with trade rule.

3. $\zeta^{(n+1)} := \zeta^{(n)} + w(\zeta^{(n)})$.

   If after signing of this contract the trajectory does not leave the limits the maximal surface then **item 4**. Otherwise the projection of this point onto maximal surface is realized: from a linear segment $[\zeta^{(n)} + w(\zeta^{(n)}), \omega]$ a point inside area limited by maximal surface is chosen at a distance no more $0.01 \times$ step from the maximal surface. Then:

   $\zeta^{(n+1)} :=$ the projected point.

4. The point is visualized onto monitor and its parameters are written into the file.

5. If the distance from $\zeta^{(n+1)}$ to equilibrium is more than $0.01 \times$ step then **item 2**. Otherwise we think that trajectory arrives at equilibrium and program stops.

Further we study the numerical example. Let preferences be defined on $\mathrm{int}\mathbb{R}^2_+$ via the following utility functions:

$$u_1(x)=47\ln(x_1)+23\ln(x_2),\;\; u_2(y)=18\ln(y_1)+54\ln(y_2),\;\; u_3(z)=17\ln(z_1)+21\ln(z_2).$$

Initial endowments are: $\omega = (\omega_1, \omega_2, \omega_3) = ((1,4),(15,3),(2,5))$. Let us also determine the following *piecewise-continuous* trade rule $w : \mathcal{A}(X) \to L^c$ which is different in each of the following areas A_1, A_2 dividing the set of all feasible allocations into two parts. In each area there is a constrain only on $x \gg 0$, and variables $y \gg 0$ and $z \gg 0$ may take any values. Define

$$A_1 = \{\zeta = (x,y,z) \geq 0 \mid x_2 \leq -0.1463x_1 + 2.6968\},$$

$$A_2 = \{\zeta = (x,y,z) \geq 0 \mid x_2 > -0.1463x_1 + 2.6968\}$$

and denote $w(\zeta) = w'(\zeta)$ when $\zeta \in A_1$ and $w(\zeta) = w''(\zeta)$ for $\zeta \in A_2$.

Further let us formally define for the area A_1 a trade rule $w' : A_1(X) \to L^c$ via the following algorithm (in A_2 the rule $w''(\zeta)$ is similarly defined).

First of all for each couple of individuals i, j, $i < j$ we construct an auxiliary vector $v^{ij}(\zeta)$ specifying some mutually beneficial exchange for the current allocation ζ. For each couple of agents this vector is constructed by some common rule; we describe it, for example, for a pair $\{1,2\}$. Let us define

$$g(\zeta) = \nabla u_1(x) + \nabla u_2(y)$$

and consider the orthogonal complement to $g(\zeta)$ in $\mathbb{R}^2$, *i.e.* a straight line defined via equation $\langle \chi, g(\zeta)\rangle = 0$, $\chi = (\chi_1, \chi_2) \in \mathbb{R}^2$. It follows from the definition $g(\zeta)$ that if gradients are *non-collinear* then for the point of the line the values $\langle \chi, \nabla u_1(x)\rangle$ and $\langle \chi, \nabla u_2(y)\rangle$ are nonzero and have *different sign*; for collinear gradients they are equal to zero. Further consider a directional vector of the line, *e.g.* it may be $(\frac{g_2}{g_1+g_2}, -\frac{g_1}{g_1+g_2}) = \bar{\chi}$ (it defines the line in a parametrical form $\chi = \gamma\bar{\chi}$, $\gamma \in \mathbb{R}$). At last define $v^{12}(\zeta) = \bar{\chi}$ if $\langle \bar{\chi}, \nabla u_1(x)\rangle > 0$ and put $v^{12}(\zeta) = -\bar{\chi}$ for $\langle \bar{\chi}, \nabla u_1(x)\rangle < 0$. Clearly that in this way we correctly define a continuous map $\zeta \to v^{12}(\zeta)$ which obeys the condition that contract $(v^{12}(\zeta), -v^{12}(\zeta))$ is mutually beneficial in non-strict form:

$$\langle v^{12}(\zeta), \nabla u_1(x)\rangle \geq 0, \quad \langle -v^{12}(\zeta), \nabla u_2(y)\rangle \geq 0.$$

Moreover if gradients are *non-collinear* then these inequalities are realized in *strict* form.

Further let us define a trade rule having appropriate properties and constructed via described maps $v^{ij}(\zeta)$.

For the area A_1 the rule is defined by formula

$$w'(x,y,z) = \beta(\zeta)\left(v^{12}(x,y) + \frac{v^{13}(x,z)}{20}, -v^{12}(x,y) + \frac{v^{23}(y,z)}{20}, \frac{-v^{13}(x,z) - v^{23}(y,z)}{20}\right),$$

and for A_2 by formula

$$w''(x,y,z) = \beta(\zeta)\left(\frac{v^{12}(x,y) + v^{13}(x,z)}{20}, v^{23}(y,z) - \frac{v^{12}(x,y)}{20}, -v^{23}(y,z) - \frac{v^{13}(x,z)}{20}\right),$$

where β is an scalar parameter chosen in an appropriate way which continuously depends on current allocation $\zeta = (x, y, z)$.

Next let us show that vectors in brackets of last expressions present mutually beneficial contracts. The fact that they are contracts is checked directly. They are also (non-strict) the mutually beneficial contracts because for $i < j$ by construction $v^{ij}(\zeta)$ we have

$$\langle \nabla u_i(\zeta), v^{ij}(\zeta)\rangle \geq 0, \quad \langle \nabla u_j(\zeta), -v^{ij}(\zeta)\rangle \geq 0,$$

that summing appropriate inequalities gives $\langle \nabla u_i(\zeta), w_i'(\zeta)\rangle \geq 0$, $i = 1, 2, 3$ and analogous thing for $w''(\zeta)$ can be obtained. Moreover, if for the current allocation there is at least *one couple of individuals, whose gradients are non-collinear* then *all* these inequalities have to be *strict*, *i.e.* they are really mutually beneficial contracts.

At last, if we shall manage to find parameter β so that to change the length of a vector specifying in our rule the "direction" of exchange (barter proportions) then we can receive the increase of individual utilities as a result of contract $w(\zeta)$ signing; *i.e.*, we need to do so that

$$u_1(x + w_1(x, y, z)) > u_1(x), \quad u_2(y + w_2(x, y, z)) > u_2(y), \quad u_3(z + w_3(x, y, z)) > u_3(z) \qquad (10.1)$$

be true for an appropriate area of definition.

Further we describe a method allowing us to find parameter β for $w'(\zeta)$ (for $w''(\zeta)$ it is done analogously). Define

$$\beta(\zeta) = \frac{1}{2}\min\{b_1, b_2, b_3\},$$

where once again $b_i = \min\{\text{step}, c_i\}$, $i = 1, 2, 3$ for the values c_i which are found in the following way:

$i = 1$, then c_1 is a solution of equation $u_1(x + c_1(v^{12}(x, y) + \frac{v^{13}(x,z)}{20})) = u_1(x)$ if it is solvable one; otherwise $c_1 = +\infty$.

$i = 2$, then c_2 is a solution of equation $u_2(y + c_2(-v^{12}(x, y) + \frac{v^{23}(y,z)}{20})) = u_2(y)$ if it is solvable one; otherwise $c_2 = +\infty$.

$i = 3$, then c_3 is a solution of equation $u_3(z + c_3(\frac{-v^{13}(x,z)-v^{23}(y,z)}{20})) = u_3(z)$ if it is solvable one; otherwise $c_3 = +\infty$.

It is easy to see, that variables $b_i(\zeta) > 0$ are correctly determined and are the continuous functions of its argument (formally one need apply the implicit function theorem and use strict concavity of utilities). Hence, $\beta(\zeta) > 0$ is also a continuous function. It has to be also clear from construction that (10.1) is fulfilled that finishes the description of a trade rule. In conclusion we note only that program parameter step > 0 appearing in construction of $b_i(\zeta)$ is used not only to define a necessary value (to do it one can take any positive number) but also to adjust the "length" of trajectory moving along a vector specifying the exchange proportions. Thus, reducing parameter *step* > 0 we approach the described discrete process to the theoretical continuous process.

For the given example the contractual trajectory constructed by the program does not converge to equilibrium. It is visible from the following Fig. 13 in which the points $\zeta^{(n)} = (x^{(n)}, y^{(n)}, z^{(n)})$ of a proper-contractual trajectory are depicted. It is curious to note, that if one applies any of rules $w'(\zeta)$, $w''(\zeta)$ as a rule for *the whole* set of feasible allocations $\mathcal{A}(X)$ then in our economy the proper-contractual UB-process is converging in computer sense...

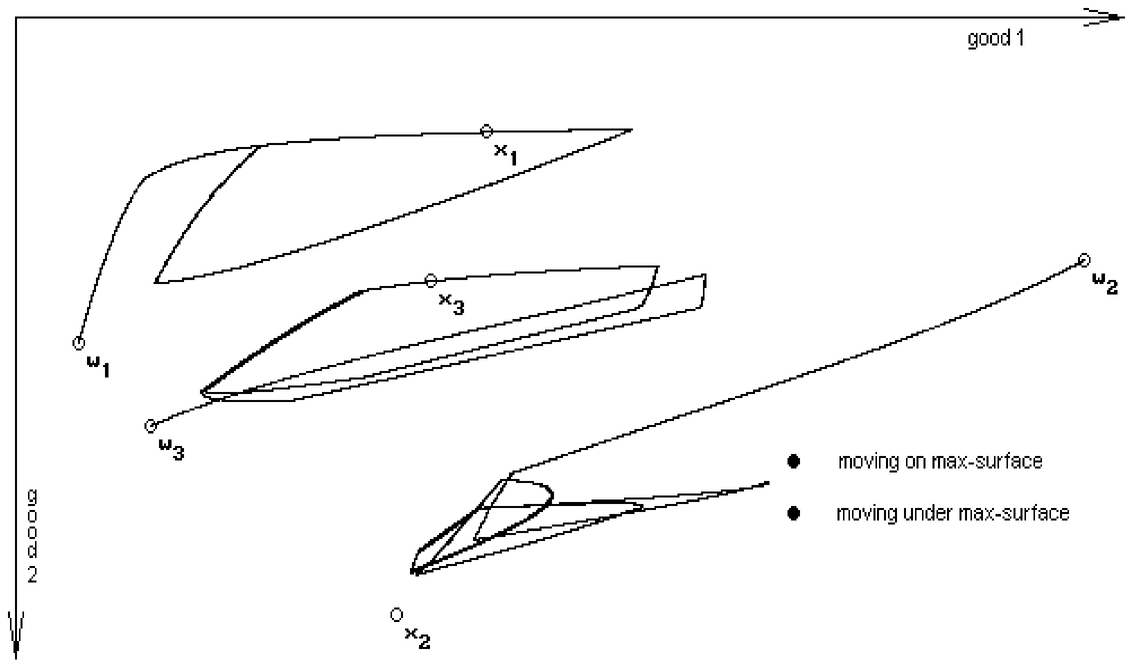

Fig. 13. Dynamics of proper contractual process with piecewise continuous trading rule in 3×2 economy

11. CONCLUSION

The present investigation includes the following results:

1. The extensive survey of literature, connected with processes, driving economic system to equilibrium. Particularly, the following processes are described and their advantages and shortcomings are discussed:

a) Walrasian tâtonnement processes driving economic system to equilibrium.

b) Processes of change of the prices, using Jacobi matrix for excess demand function.

c) Disequilibrium models of trade processes.

d) Edgeworth's processes.

e) Strategic approach.

General volume of the quoted here literature has made about 30 items.

2. Contractual processes are in detail described and, first of all, proper-contractual ones — they are processes, in which partial breaking of the contracts is allowed. With this in mind several basic hypotheses, determining the character of contracts' breaking process, are formulated in a general kind and for major particular cases. They are the following:

(IB) — instantaneous breaking of the contracts;

(UB) — uniform breaking of all contracts;

(CUB) — uniform breaking of gross within-coalitional contracts.

Combinations of these hypotheses result to proper-contractual trajectories of a different kind of a generality. Under **(IB)** and **(UB)** contractual trajectory turns out *aggregated*, Under **(IB)** and **(CUB)** — *coalitional-contractual*; there are given formal and mathematically reasonable definitions. In my opinion, the coalitional-contractual trajectory should serve the central concept in further researches. Besides, there is described specific contractual process, adequate a case of the pairwise bargains and to simultaneous breaking of contracts not only in active, but also in passive coalitions. At last, concept of *trade rule* is introduced; this is a map, unequivocally determining mutually beneficial contract for the current consumption plans, having some additional good mathematical properties. By use of a trade rule a contractual trajectory of each mentioned kinds is unequivocally determined. The special type of *benevolent* rules of trade is stood out as rules which determine a new contract allowing the break of gross barter contract only if being realized *every* new mutually beneficial contract involves contracts' breaking. Just for this class of benevolent processes the basic positive results about convergence were received.

3. For considered kinds of proper-contractual trajectories a variant of parallel price process is offered. In this process the current prices are determined as an average (in specific sense) vector of exchange proportions under all bargains, really carried out for the current time moment.

4. Presented analysis of convergence of contractual trajectories has given the following results:

a) For the economies with 2 individuals and 2 commodities convergence of proper contractual processes relative to any continuous trading rule has proven under rather general assumptions. Local stability of equilibria was also investigated and a reasonable criterion for this was suggested.

b) The theorem on convergence to equilibrium of non-degenerate benevolent UB-contractual processes has been proven. In addition, local stability of equilibria relative to benevolent trading rules was analyzed; however appropriate theorem was proven only for 2 agents economy with an arbitrary finite commodity space.

c) A series of model examples are presented demonstrating specific properties of contractual processes in different cases. There are examples for converged process and also two examples where the contractual process is cycling. First of them presents economy with 4 agents and 2 commodities and coalitional contractual process (CUB). In second example for economy with 3 agents UB-process with *piecewise continuous* trading rule is cycling.

APPENDICES

A1. List of notations and special symbols

Let C be a subset of topological vector space, then:

$\operatorname{int} C$ denotes the interior of C,

$\operatorname{ri} C$ (relative interior) denotes the interior of C relative to its affine hull $\operatorname{aff} C$,

$\operatorname{co} C$ denotes convex hull,

$\operatorname{cl} C$ denotes the closure of C.

$E = \mathbb{R}^l$ is l-dimension commodity space.

$\mathcal{I} = \{1, \ldots, n\}$ is the set of agents.

$L = E^n = E^{\mathcal{I}}$ is the space of economy allocations.

$L^c = \{(v_1, \ldots, v_n) \in E^{\mathcal{I}} \mid \sum_{\mathcal{I}} v_i = 0\}$ is the space of contracts.

$X_i = E_+ = \mathbb{R}^l_+$ is consumption set of agent $i \in \mathcal{I}$.

$X = \prod_{i \in \mathcal{I}} X_i$.

$\omega_i \in X_i$ is the vector of ith agent initial endowments.

$\omega = (\omega_i)_{i \in \mathcal{I}} \in X$ is the vector of initial endowments of all traders of the economy.

$u_i : X_i \to \mathbb{R}$ is i's agent utility function.

$\mathcal{A}(X) = \{x = (x_i)_{i \in \mathcal{I}} \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \omega_i\}$ is the set of all *feasible allocations*.

$\{x_i \in X_i \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0\}$ is the maximal surface of agent $i \in \mathcal{I}$.

$h_i(x_i) = \nabla u_i(x_i) + \nabla^2 u_i(x_i)(x_i - \omega_i)$ is the normal vector for tangent hyperplane to i's maximal surface at the point $x_i \in X_i$.

$\mathcal{I}^a(x) = \{i \in \mathcal{I} \mid \langle \nabla u_i(x_i), x_i - \omega_i \rangle = 0\}$ is the set of all active individuals at $x \in \mathcal{A}(X)$.

$W^{fr}(x) = \{w \in L^c \mid \langle \nabla u_i(x_i), w_i \rangle > 0, \ \forall i \in \mathcal{I} \ \& \ \langle h_i, w_i \rangle > 0, \ \forall i \in \mathcal{I}^a(x)\}$ is the set (possible empty) of all mutually beneficial contracts that being signed do not attract the break of aggregated contract $x - \omega$.

A2. Proofs

Proof of Lemma 7.1. So let alternative (**i**) be fulfilled and let starting at the moment $\tau \geq 0$, only 1st individual can be active and second is passive for all $t \geq \tau$. Further, let $\tilde{x}_1$ (1st agent bundle) be any limit point of trajectory. Define $\tilde{x}_2 = \omega_1 + \omega_2 - \tilde{x}_1$ and show that allocation $(\tilde{x}_1, \tilde{x}_2)$ is Pareto optimal. Assuming contrary due to trade rule definition we conclude $\langle \nabla u_1(\tilde{x}_1), v_1(\tilde{x}_1) \rangle > 0$ and by continuity this property has to be fulfilled in some neighborhood of the point $\tilde{x}_1$, *i.e.*

$$\exists \ \varepsilon > 0 : \quad \langle \nabla u_1(x_1), v_1(x_1) \rangle > 0, \quad \forall \, x_1 \in B_{2\varepsilon}(\tilde{x}_1),$$

where $B_{2\varepsilon}(\tilde{x}_1)$ is closed ball with the radius 2ε centered at the point $\tilde{x}_1$. Since the ball is a compact set and by continuity it is equivalent to

$$\exists \ \varepsilon > 0, \ \delta > 0 : \quad \langle \nabla u_1(x_1), v_1(x_1) \rangle > \delta, \quad \forall \, x_1 \in B_{2\varepsilon}(\tilde{x}_1). \tag{11.1}$$

We observe from this that one can come to a contradiction if one manages to show that the current point of trajectory is in the ball during infinite (by measure) time.

Really, since 2nd agent is passive (almost everywhere) and because 1st agent utility is increasing monotonically along the trajectory (starting at the moment τ) the following estimations are fulfilled:

$$u_1(x_1(t)) - u_1(x_1(\tau)) = \int_\tau^t \frac{du_1(x_1(\zeta))}{d\zeta} d\zeta = \int_\tau^t \langle \nabla u_1(x_1(\zeta)), \dot{x}_1(\zeta) \rangle d\zeta \geq$$

$$\geq \int_\tau^t \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d\zeta \geq \int_{[\tau,t]\cap\Omega} \langle \nabla u_1(x_1(\zeta)), v_1(x_1(\zeta)) \rangle d\zeta \geq \delta \cdot \mu([\tau, t] \cap \Omega).$$

Here $\Omega \subset [\tau, +\infty]$ is the set of all time moments when a current point of trajectory $x_1(\zeta)$ is located in the ball $B_{2\varepsilon}(\tilde{x}_1)$ and $\mu([\tau, t] \cap \Omega)$ is Lebesgue measure of the set $[\tau, t] \cap \Omega$. If $\mu(\Omega) = +\infty$ we have $\mu([\tau, t] \cap \Omega) \to +\infty$ for $t \to +\infty$. Then due to the last estimation it has to be $u_1(x_1(t)) \to +\infty$ that is impossible since the set of all allocation is compact and utility function is continuous.

Let us show that $\mu(\Omega) = +\infty$. It is obvious if starting at some time moment $t \geq \tau$ all points of trajectory are located in the ball. In the contrary case one can find an enumerable set of moments t_k, t'_k, $k = 1, 2 \ldots$ such that $\| x_1(t_k) - \tilde{x}_1 \| < \varepsilon$ and $t'_k > t_k$ is a *closest* after t_k time moment when the trajectory leaves the ball, *i.e.*

$$\|x_1(t'_k) - \tilde{x}_1\| = 2\varepsilon \;\; \& \;\; \|x_1(\zeta) - \tilde{x}_1\| < 2\varepsilon, \; \forall\, \zeta \in [t_k, t'_k).$$

However in such a case we have an estimation:

$$\varepsilon \leq \|x_1(t'_k) - x_1(t_k)\| = \| \int_{t_k}^{t'_k} \dot{x}_1(\zeta) d\zeta \| \leq \int_{t_k}^{t'_k} \|\dot{x}_1(\zeta)\| d\zeta \leq c \int_{t_k}^{t'_k} d\zeta = c(t'_k - t_k),$$

where $c > 0$ is an upper bound for the norm of right hand part of the law (4.8), *i.e.* this is a value satisfying

$$c \geq \|\lambda^{min}(x, v_1)(x_1 - \omega_1) + v_1(x_1)\|, \quad \forall\, x_1 \in B_{2\varepsilon}(\tilde{x}_1).$$

Due to imposed assumptions and from the compactness and continuity of objects that we need it is easy to prove that the right hand part of this inequality is bounded from above and, therefore, such $c > 0$ does exist. As a result we have got the estimation

$$(t'_k - t_k) \geq \frac{\varepsilon}{c} > 0, \quad \forall\, k = 1, 2, \ldots$$

Moreover via construction all intervals $[t_k, t'_k]$ are pairwise non-intersected and $[t_k, t'_k] \subset \Omega$, $\forall\, k = 1, 2, \ldots$ Therefore, $\mu(\Omega) = +\infty$. Thus we obtain a contradiction that proves Pareto optimality of the allocation under study.

To state the second part of lemma remember that every allocation from the interior of direct product of consumption sets which is Pareto optimal and simultaneously stable relative to the partial break of gross contract is an equilibrium, see Theorem 2.2 from Marakulin (2003).

Proof of Lemma 7.2. Let τ' be some time moment when first agent is active. Now define a time moment when 1st agent is active $\tau_1^1 \geq \tau'$ and such that it is earlier of *first* moment $\tau'' > t'$ when

2nd agent is active and such that on the interval (τ_1^1, τ'') both agents are passive. Here τ_1^1 is the *latest* moment of 1st agent activity on the interval $[\tau', \tau'')$. Analogously, for 2nd agent one can find a moment τ_1^2 as a moment of last his/her activity up to the nearest moment $\tau''' > \tau''$ when 1st agent is active. In view of compactness and continuity of objects under study all considered time moments do exist. For example, τ'' and τ_1^1 can be found by formulas

$$\tau'' = \min\{t \in [\tau', +\infty) \mid \langle \nabla u_2(x_2(t)), x_2(t) - \omega_2 \rangle = 0\},$$

$$\tau_1^1 = \max\{t \in [\tau', \tau''] \mid \langle \nabla u_1(x_1(t)), x_1(t) - \omega_1 \rangle = 0\}.$$

Further taking the point τ''' as "initial" (*i.e.* instead of τ') in the described above procedure, one can find the moments τ_2^1 and τ_2^2, accordingly. Show that constructed in this way fragments of sequences that we need to be found obey the requirement (7.1). With this in mind first let us better understand the geometry of moving of a trajectory and reveal some peculiarities of this moving.

Really by construction on intervals $[\tau_1^1, \tau'']$ and $[\tau_1^2, \tau_2^1]$ the 1st agent utility increases: it is so because only mutually beneficial contracts are signed during contractual process and also because in our intervals 2nd agent is passive. It has to be shown that for all points t from interval $[\tau'', \tau_1^2]$ the inequality $u_1(x_1(t)) > u_1(x_1(\tau_1^1))$ is fulfilled. Let us do it.

Now consider the moment τ''. By construction the following relations

$$\langle \nabla u_1(x_1(\tau'')), v_1(x_1(\tau'')) \rangle > 0, \qquad \langle \nabla u_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,$$

$$\langle \nabla u_2(x_2(\tau'')), x_1(\tau'') - \omega_1 \rangle = 0$$

have to be true. Moreover if $h_2(x_2(\tau'')) = \nabla u_2(x_2(\tau'')) - \nabla^2 u_2(x_2(\tau''))(x_1(\tau'') - \omega_1)$ satisfies[36]

$$\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle < 0,$$

then the condition (4.5) of contracts break is violated and it means that a trajectory only "touches" with maximal surface at the point $x_1(\tau'')$ and then "leaves" it. Therefore in a neighborhood of the moment τ'' a break of contracts does not occur and both utilities are locally increased. A break of contracts may occur only if

$$\langle h_2(x_2(\tau'')), v_1(x_1(\tau'')) \rangle \geq 0$$

and if for small $\Delta t > 0$ at the points $\tau'' + \Delta t$ this inequality is strict. Thus after the "going through" the point $x_1(\tau'')$ a trajectory $x_1(t)$ will move some non-zero time in framework of ε-extension of a cone with the vertex at the point $x_1(\tau'')$ which is defined by inequalities:

$$\langle h_2(x_2(\tau'')), x_1 \rangle \geq \langle h_2(x_2(\tau'')), x_1(\tau'') \rangle,$$

$$\langle \nabla u_2(x_2(\tau'')), x_1 \rangle \leq \langle \nabla u_2(x_2(\tau'')), x_1(\tau'') \rangle = \langle \nabla u_2(x_2(\tau'')), \omega_1 \rangle.$$

More exactly, due to $\langle h_2(x_2(t)), \dot{x}_2(t) \rangle = 0$, see (4.2), a limit deviation of trajectory will be realized along an edge of this cone, see Fig. 5.

Further, on interval $[\tau_1^1, \tau'']$ in the plane a trajectory $x_1(t)$ circumscribes an continuous curve with the ends $x_1(\tau_1^1)$ and $x_1(\tau'')$ such that for $t \in (\tau_1^1, \tau'')$ the points $x_1(t)$ are located strictly

[36] Remember that $x_2(\tau'') - \omega_2 = -(x_1(\tau'') - \omega_1)$ and $v_2(x_2(\tau'')) = -v_1(x_1(\tau''))$.

"below" than a point of maximal surface being intersected with the ray starting from ω_1 and going through the point $x_1(t)$, because $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i\rangle > 0$, $i = 1, 2$. Moreover for $t < \tau''$ and *close* to τ'' it has to be

$$\langle \nabla u_2(x_2(\tau'')), x_1(\tau'')\rangle < \langle \nabla u_2(x_2(\tau'')), x_1(t)\rangle \iff \langle \nabla u_2(x_2(\tau'')), \frac{x_1(t) - x_1(\tau'')}{\tau'' - t}\rangle > 0,$$

because $v_2(x_1(\tau'')) = -v_1(x_1(\tau'')) \approx \frac{x_1(t)-x_1(\tau'')}{\tau''-t}$. Thus there exists a moment $t' < \tau''$ such that $x_1(t') - \omega_1 = \gamma(x_1(\tau'') - \omega_1)$ for some $0 < \gamma < 1$ and, simultaneously, all points of trajectory from interval $t \in [\tau_1^1, t']$ obey $\langle \nabla u_2(x_2(\tau'')), x_1(t)\rangle < \langle \nabla u_2(x_2(\tau'')), \omega_1\rangle$. The similar inequality has to be fulfilled for the points $x_1(t)$, $t \in [\tau'', \tau_1^2]$ because contracts are mutually beneficial and 1st agent is passive on this time interval.

Further we are going to the final part of the proof. Assume that for some $t \in [\tau'', \tau_1^2]$ the inequality $u_1(x_1(t)) \le u_1(x_1(\tau_1^1))$ is fulfilled. Now from the continuity and due to presented above reasonings it follows that there are moments $t'' < \tau''$ and $\tau'' < t''' \le \tau_1^2$ such that $x_1(t'') = x_1(t''')$ is true. Consider the first possible moment of this type (one needs to take minimal t''' having this property). For $t'' > \tau_1^1$ we have a contradiction since then our trajectory is cycling (due to the law of change is autonomous) and never arrives to a point on 1st agent maximal surface but it has to be so at the moment $\tau_2^1 > t'''$. Therefore, it has to be $t'' = \tau_1^1$. However $x_1(\tau_1^1)$ is a point on 1st agent maximal surface where 2nd agent is passive. Hence there is a neighborhood of $x_1(\tau_1^1)$ such that 1st agent utility strictly increases along *every* trajectory starting from any point from the neighborhood. Therefore for all small enough $\varepsilon > 0$ it has to be $u_1(x_1(t''' - \varepsilon)) < u_1(x_1(t'''))$. Moreover for some $\varepsilon > 0$ no point $x_1(t)$, $t \in (t''' - \varepsilon, t''')$ can be located on 2nd agent maximal surface (otherwise at the point $x_1(t''') = x(\tau_1^1)$ both individuals are active that is possible only at an equilibrium which trajectory can never leave). Therefore the last moment of trajectory being on 2nd agent maximal surface, by definition this is the moment τ_1^2, has to be realized *earlier* the moment t''' because the point $x_1(t''') = x_1(t'') = x(\tau_1^1)$ is located on 1st agent maximal surface. Thus, it has to be $t''' > \tau_1^2$ but this is impossible. The obtained contradictions finish the proof of Lemma 7.2.

Proof of Proposition 8.1. Let $V_{\bar x}$ be a neighborhood of equilibrium point $\bar x = (\bar x_1, \bar x_2)$ which existence is postulated in Proposition 8.1. Consider a vector-function $f : V_{\bar x} \to \mathbb{R}^2$ where

$$f_i(y_i) = \min\{u_i(y_i), u_i(\bar x_i)\}, \quad i = 1, 2.$$

Show that this function is (non-strictly) monotonically increasing along benevolent contractual trajectory.

In fact, for $x(t) \in V_{\bar x}$ either $\mathcal{I}^a(x(t)) = \emptyset$ but then at the point $x(t)$ contractual process is going without break and monotonicity is obvious, or $\mathcal{I}^a(x(t)) \ne \emptyset$. In the last case if $W^{fr}(x(t)) \ne \emptyset$ then monotonicity follows from (8.2), in a contrary case $W^{fr}(x(t)) = \emptyset$ and one can apply (8.3). Further we first note that $(u_1(x_1(t)), u_2(x_2(t))) < (u_1(\bar x_1), u_2(\bar x_2))$ is impossible because otherwise the equilibrium $\bar x$ Pareto dominates $x = x(t)$ and $W^{fr}(x(t)) \ne \emptyset$. Therefore, since by (8.3) it has to be $u_j(x_j(t)) \le u_j(\bar x_j)$, $j \in \mathcal{I}^a(x(t))$ and then $u_i(x_i(t)) > u_i(\bar x_i)$ for $i \ne j$. However in this case by specification $f_i(x_i(t')) = f_i(\bar x_i)$ for all $t' \ge t$ close enough to t. The activity of j also implies that the function $f_j(x_j(t'))$ is locally increasing for $t' \ge t$.

Further, due to assumption that utility functions are strictly concave and since equilibrium allocation is Pareto optimal it is not difficult to prove that the sets

$$V_{\bar x}^{\varepsilon} = \{y \in \mathcal{A}(X) \mid f_1(y_1) \ge f_1(\bar x_1) - \varepsilon \ \ \& \ \ f_2(y_2) \ge f_2(\bar x_2) - \varepsilon\}, \quad \varepsilon > 0$$

form a basis of neighborhoods for the point $\bar{x}$ in $\mathcal{A}(X)$.[37] Really to see this it is enough to note that $\cap_{\varepsilon>0} V^{\varepsilon}_{\bar{x}} = \{\bar{x}\}$. However this is true because only Pareto optimal points can be in the intersection but for strictly concave utility functions it is impossible to find two different allocations which are Pareto optimal and simultaneously have equal agents' utilities. Choosing now $\varepsilon > 0$ from condition $V^{\varepsilon}_{\bar{x}} \subset V_{\bar{x}}$ and taking $V^{\varepsilon}_{\bar{x}}$ as a neighborhood of initial data, we conclude that the first condition of local stability (see above) is valid: a trajectory being at least one time in this neighborhood can never leave it (due to the monotonicity of f along a benevolent trajectory).

Further, the monotonicity of f along a trajectory and the property $f(x(t)) \le f(\bar{x})$ allows us to consider f as a Lyapunov vector-function but we need to show that $\lim_{t\to\infty} x(t) = \bar{x}$. Let us do it. Let $\tilde{x}$ be any limit point of trajectory $x(t)$. The monotonicity of f implies that $\lim_{t\to\infty} f(x(t))$ does exist and also $\lim_{t\to\infty} f(x(t)) = f(\tilde{x}) \le (u_1(\bar{x}_1), u_2(\bar{x}_2))$. Show that this inequality can be fulfilled only as equality. To do it first of all notice that the case $f(\tilde{x}) \ll (u_1(\bar{x}_1), u_2(\bar{x}_2))$ is obviously impossible (by Lemma 7.1 and Remark 7.1). Hence at least for one component equality is realized. Let, for example, for 1st agent and for all large enough t it is realized: $\exists\, \delta > 0$: $u_1(x_1(t)) < u_1(\bar{x}_1) - \delta$. Now (8.3) and benevolence imply that the utility of this agent is monotonically increased for all t large enough. Moreover it is easy to see that if $u_2(\tilde{x}_2) > u_2(\bar{x}_2)$ then we are in condition of alternative (**i**) from the previous paragraph § 7.2 (see page 44) because only 1st agent can be active for t large enough. Therefore due to Lemma 7.1 and Remark 7.1 the allocation $\tilde{x}$ is Pareto optimal. However in so doing $\tilde{x}$ has to be stable relative to partial break of gross contract $\tilde{x} - \omega$. Hence $\tilde{x}$ has to be an equilibrium allocation from the neighborhood $V^{\varepsilon}_{\bar{x}}$. Choosing now $\varepsilon > 0$ so that the neighborhood does not incudes equilibria different from $\bar{x}$ and taking this neighborhood as a neighborhood for initial data we conclude that $\tilde{x} = \bar{x}$.

Thus we have proven that $(u_1(\tilde{x}_1), u_2(\tilde{x}_2)) \ge (u_1(\bar{x}_1), u_2(\bar{x}_2))$. However once again it means that $\tilde{x}$ is Pareto optimal and therefore it is an equilibrium. Hence $\tilde{x} = \bar{x}$. As a result: we have found a neighborhood such that any trajectory defined by benevolent contractual process which is going through some point of the neighborhood has all limits points equal to $\bar{x}$. This trajectory converges to $\bar{x}$.

Proof of Lemma 9.3. Fist let us choose $\varepsilon > 0$ so that if for $j \in \mathcal{I}$ inequality $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \omega_j \rangle > 0$ is fulfilled then for all $t \in (\tau, \tau + \varepsilon)$ the similar inequality $\langle \nabla u_j(x_j(t)), x_j(t) - \omega_j \rangle > 0$ is also fulfilled. It is possible in view of continuous dependence of a trajectory from time and sice all functions participating in an inequality are continuous.

Further, let $\langle \nabla u_j(x_j(\tau)), x_j(\tau) - \omega_j \rangle = 0$ be fulfilled for some $j \in \mathcal{I}$, $j \ne i$, *i.e.*, j is another agent distinct from i which is active at the moment τ. Applying Definition 9.1 suppose, for example, that

$$\frac{\langle h_i(x_i(\tau)), v_i(x_i(\tau)) \rangle}{\langle h_i(x_i(\tau)), \omega_i - x_i(\tau) \rangle} < \frac{\langle h_j(x_j(\tau)), v_j(x_j(\tau)) \rangle}{\langle h_j(x_j(\tau)), \omega_j - x_j(\tau) \rangle}$$

holds. From a continuity of functions participating in the inequality it is possible also to find a neighborhood of point $x(\tau)$ in $\mathcal{A}(X)$ and a neighborhood of point $v(x(\tau))$ in the space of contracts L^c such that the similar inequality is true for any point from these neighborhoods

[37] It means that all these sets are neighborhoods and that every neighborhood includes a set of this type.

replacing $x(\tau)$ and $v(x(\tau))$, accordingly. Let $\delta > 0$ be such that

$$\frac{\langle h_i(x_i), w_i\rangle}{\langle h_i(x_i), \omega_i - x_i\rangle} < \frac{\langle h_j(x_j, w_j\rangle}{\langle h_j(x_j), \omega_j - x_j\rangle}, \quad \forall\, x \in B_\delta(x(\tau)) \cap \mathcal{A}(X),\ \forall\, w \in B_\delta(v(x(\tau))) \cap L^c$$

is fulfilled, where $B_\delta(y)$ denotes a ball of radius $\delta > 0$ centered at y in an appropriate space, and vectors $h_i(x_i)$, $h_j(x_j)$ are formally defined by formula (4.2) and are calculated at the designated point of space. Moreover, without lost of generality it is possible also to think that *all contracts* from $B_\delta(v(x(\tau))) \cap L^c$ *are beneficial* at every point from $B_\delta(x(\tau)) \cap \mathcal{A}(X)$. This obviously follows from the definition of the mutually beneficial contract and from the continuity of all functions participating in required inequalities. Besides it is possible to think that *numerator and denominator* in expressions from the last formula *do not change a sign* for all points of chosen neighborhoods and that this is true for any pair of active individuals at the moment τ. At last, reducing if necessary, $\varepsilon > 0$ can be chosen so that all points $x(t)$ for $t \in (\tau, \tau + \varepsilon)$ are in the limits of the chosen neighborhood $B_\delta(x(\tau))$ of $x(\tau)$, *i.e.* for the time not more than $\varepsilon > 0$ the trajectory does not leave this neighborhood.

Further let us establish the validity of alternative (**i**). It is necessary to show that in conditions of (**i**) an arbitrarily chosen point of the trajectory $x(t)$, $t \in (\tau, \tau + \varepsilon)$ is located on the maximal surface of individual i, *i.e.*, that $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i\rangle = 0$ is fulfilled.

Assuming $\langle \nabla u_i(x_i(t)), x_i(t) - \omega_i\rangle > 0$ find a maximum of all those moments $t' \in [\tau, t)$ where the current point of trajectory $x(t')$ is located on the maximal surface of agent i. As $x(\tau)$ is on the maximal surface of the agent i, this maximum does exist and obviously that at this moment the point of a trajectory is located on the maximal surface. Let s denote this maximum. Now we have $\langle \nabla u_i(x_i(s)), x_i(s) - \omega_i\rangle = 0$ and $\langle \nabla u_i(x_i(\zeta)), x_i(\zeta) - \omega_i\rangle > 0$ for all $\zeta \in (s, t)$. By definition in the interval (s, t) the law of change of a trajectory (4.8) is set by the contract $v(x(\zeta))$ and by (in general discontinuous) function $\lambda^{min}(\cdot)$ which in conditions of alternative (**i**) by the choice of ε and because individual i is passive for all $\zeta \in (s, t)$ has to satisfy

$$\lambda^{min}(x(\zeta), v(x(\zeta))) > a > b > \frac{\langle h_i(x_i(\zeta)), v_i(x(\zeta))\rangle}{\langle h_i(x_i(\zeta)), \omega_i - x_i(\zeta)\rangle} = g_i(x(\zeta)), v(x(\zeta)) \tag{11.2}$$

for some real a, b. Further the vector $h_i(x_i)$ participating the the right hand part of inequality (11.2) is defined by formula (4.2) and, therefore, it is the gradient of function $F(x_i) = \langle \nabla u_i(x_i), x_i - \omega_i\rangle$ which defines maximal surface by equation $F(x_i) = 0$. So we have $F(x_i(s)) = 0$ and the value $F(x_i(t))$ can be found by formula

$$F(x_i(t)) = \int_s^t \frac{F(x_i(\zeta))}{d\zeta} d\zeta = \int_s^t \langle \nabla_{x_i} F(x_i(\zeta)), \dot{x}_i(\zeta)\rangle d\zeta,$$

where the function under integral is summarized (since $x(\cdot)$ is an absolute continuous function). Substituting expression of under-integral functions ($\dot{x}_i(\zeta)$ via the law of trajectory) in view of (11.2) and $\langle h_i(x_i(\zeta)), x_i(\zeta) - \omega_i\rangle < 0$, $\forall\, \zeta \in (s, t)$ we obtain the following estimation

$$F(x_i(t)) = \int_s^t \langle h_i(x_i(\zeta)), \lambda^{min}(\zeta)(x_i(\zeta) - \omega_i) + v_i(x_i(\zeta))\rangle d\zeta \leq$$

$$\leq a \int_s^t \langle h_i(x_i(\zeta)), (x_i(\zeta) - \omega_i)\rangle d\zeta + \int_s^t \leq \langle h_i(x_i(\zeta)), v_i(x_i(\zeta))\rangle d\zeta \leq$$

$$\leq (a-b)\int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta)-\omega_i\rangle d\zeta + \int_s^t \langle h_i(x_i(\zeta)), g_i(\zeta)(x_i(\zeta)-\omega_i)+v_i(x_i(\zeta))\rangle d\zeta =$$

$$= (a-b)\int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta)-\omega_i\rangle d\zeta.$$

The last equality in the chain of estimations is true because the second integral (summand) is equal to zero: by definition of $g_i(\zeta) = g_i(x(\zeta), v(x(\zeta))$ in (11.2) and due to

$$\langle h_i, \frac{\langle h_i, v_i\rangle}{\langle h_i, \omega_i - x_i\rangle}(x_i-\omega_i)+v_i\rangle = 0, \quad x_i \neq \omega_i.$$

Since $\int_s^t \langle h_i(x_i(\zeta)), x_i(\zeta)-\omega_i\rangle d\zeta < 0$ then as a result we conclude

$$F(x_i(t)) = \langle \nabla u_i(x_i(t)), x_i(t)-\omega_i\rangle < 0,$$

that contradicts the initial assumption. Thus alternative (**i**) has proven.

In a part of the proof of alternatives (**ii**) and (**iii**) we only note that it can be done in accordance with the same method as stated above. The difference consists in the formulation of an requirement similar to (11.2) but written down concerning other parameters: only this thing is important to obtain the key estimations. For example, for the proof of alternative (**ii**), for the individual $j \neq i$ which is active at the moment τ for a suitable time interval one needs to apply

$$\lambda^{min}(x(\zeta), v(x(\zeta))) < a < b < \frac{\langle h_j(x_j(\zeta)), v_j(x(\zeta))\rangle}{\langle h_j(x_j(\zeta)), \omega_j - x_j(\zeta)\rangle} = g_j(x(\zeta)), v(x(\zeta)).$$

Lemma 9.3 has proven.

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